Exercises for Math 276

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These exercises will be updated periodically throughout the course of the semester.

You do not have to attempt all the exercises. They are here to help cement the concepts discussed in lectures and to clarify some points made in lectures.

Please inform me if you find any errors or have any questions about the exercises.

1 Exercises due Feb 21st

1. Let S be a set. The *free Abelian group* on S is a pair $(F(S), i_S)$ consisting of an Abelian group $F(S)$ and an function $i_S\colon S \to F(S)$ such that for any Abelian group A and map $j\colon S \to A$ there is a unique homomorphism $\phi\colon F(S) \to A$ such that $\phi \circ i_S = j$.
   (a) Find a map $i_S\colon S \to \bigoplus_S \mathbb{Z}$, such that the above property holds.
   (b) Suppose that $(F(S), i_S)$ and $(F(S)', i'_S)$ are free abelian groups on S. Show that there is an isomorphism $\phi\colon F(S) \to F(S)'$ such that $\phi \circ i_S = i'_S$.
   (c) Show that we can extend the function $S \mapsto F(S)$ to a functor Set$\to$Ab.
   (d) Show that $X \mapsto C_n(X)$ can be extended to a functor Top$\to$Ab.
2. Let $\sigma\colon I \to X$. Show that $\bar{\sigma} \sim -\sigma$. Where $\bar{\sigma}(t) = \sigma(1-t)$.
3. Let A be a subspace of X and $i\colon A \to X$ the inclusion map. Show that i extends to an injective homomorpshism $C_n(A) \to C_n(X)$.
4. Suppose $r\colon X \to A$ is a retraction. I.e. $r \circ i\colon A \to A$ is the identity.
   (a) Show that $r_*\colon H_n(X) \to H_n(A)$ is surjective.
   (b) Show that $i_*\colon H_n(A) \to H_n(X)$ is injective.
   (c) Find spaces A, X such that $A \subset X$ and for some n $i_*\colon H_n(A) \to H_n(X)$ is not injective. (Hint: look at the case $n = 0$.)
5. (optional) Let S be obtained by taking a disjoint union of 2-simplices and identifying edges in pairs, where every edge is in a unique pair. Show that S is locally homeomorphic to $\mathbb{R}^2$.
6. (a) Show that if two loops are homotopic relative to $\{0, 1\}$, then they are homologous.
   (b) Deduce that there is a homomorphism $\pi_1(X, b) \to H_1(X)$.
   (c) Following the ideas from class show that if X is path connected, then the map above is surjective.

7. (optional) Let V be a totally ordered finite set. Let $L = (V, \Sigma)$ be a simplicial complex. Let $S_n(L)$ be the free abelian group on the set $\{[v_0, \ldots, v_n] \mid v_i \in V, \{v_0, \ldots, v_n\} \in \Sigma, v_i < v_j \text{ if } i < j\}$. Let $\delta_n : S_n(L) \to S_{n-1}(L)$ be given by $\delta_n([v_0, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v_i}, \ldots, v_n]$.

   (a) Show that $\delta_n \circ \delta_{n+1} = 0$.

   (b) Define the n-th simplicial homology group $H_n^{\Delta}(L) = \ker(\delta_n)/\mathrm{Im}(\delta_{n+1})$. Show that if L is an n-dimensional simplicial complex then $H_k^{\Delta}(L) = 0$ for all $k > n$.

   (c) Show that if L is an n-dimensional simplicial show that $H_n(L)$ is free-abelian.

   (d) Pick a triangulation for S^1 and S^2, compute their simplicial homology.

   It is worth noting that simplicial homology is isomorphic to singular homology. A proof can be found in Hatcher, we will not study it in this courses for the issues mentioned in lectures.

2 Exercises due 28th Feb

1. Show that any long exact sequence can be broken up into short exact sequences.

2. Let $D_n(X, A)$ be the free abelian group on the singular n-simplices σ such that $\mathrm{Im}(\sigma) \not\subset A$.

   (a) Show that $C_n(X) = C_n(A) + D_n(X, A)$.

   (b) Show that $C_n(A) \cap D_n(X, A) = 0$.

   (c) Show that $D_n(X, A) \cong C_n(X, A)$. The isomorphism is the restriction of the projection $C_n(X) \to C_n(X, A)$.

   (d) Deduce that the short exact sequence $0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$ splits.

   (e) Why isn't this splitting a chain map?

3. Finish the proof of the snake lemma by showing that $i_* \delta_* = 0$ and the sequence is exact at both $H_n(A_*)$ and $H_n(C_*)$.

4. Show that if C is free, then the short exact sequence $0 \to A \to B \to C \to 0$ splits.

5. Show that the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ does not split.

6. (optional) Show that if a short exact sequences $0 \to A \to B \to C \to 0$ splits. Then $B \cong C \bigoplus A$.

7. (optional) Let G be an Abelian group. Define $C_n(X; G) = \{\sum_\sigma g_\sigma \sigma \mid \sigma : \Delta^n \to X, g_\sigma \in G\}$. This is a group with pointwise addition and is isomorphic to $\bigoplus_\sigma G$. Define $\partial : C_n(X; G) \to C_{n-1}(X; G)$ as for singular homology. Note $-g$ still makes sense since G is an Abelian group. We define the homology of X with coefficients in G to be $H_n(X; G) = H_n(C_*(X; G))$.

   (a) Show that $H_n(pt; G) = G$ if $n = 0$ and is trivial otherwise.

   (b) Show that if $0 \to G' \to G \to G'' \to 0$ is a short exact sequence of Abelian groups. Then the induced sequence $0 \to C_n(X; G') \to C_n(X; G) \to C_n(X; G'') \to 0$ is also exact.

   (c) Deduce that there is a long exact sequence in homology with coefficients.

   In the special case that $G' = G = \mathbb{Z}$ and $G'' = \mathbb{Z}/p\mathbb{Z}$ the connecting homomorphism is known as the Bockstein homomorphism.

8. (optional)

(a) Show that the fundamental group of the Klein bottle $G = \langle x, y \mid x^{-1}yx = y^{-1} \rangle$ fits into a short exact sequence $0 \to \mathbb{Z} \to G \to \mathbb{Z} \to 0$.

(b) Show that the above sequence splits.

(c) Show that G is not Abelian.

(d) Deduce that for non-abelian splitting does not mean $G = \mathbb{Z} \times \mathbb{Z}$.

(e) Look up the definition of a semi-direct product to see what happens in the non-Abelian case.

3 Exercises due March 6th

1. (a) Show that there is no retraction from $D^n \to S^{n-1}$.

   (b) Show that every map from $D^n \to D^n$ has a fixed point.

2. (a) Show that S^n is not homotopy equivalent to S^m if $n \neq m$.

   (b) Show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq m$.

3. Finish the proof of the 5-lemma by showing that the map f_3 is surjective.

4. (optional) Let $A \subset B \subset C \subset X$ be topological spaces. Assume B is open and $\bar{B} \subset \text{int}(C)$.

   Assume that there is a deformation retraction $r\colon C \to A$, i.e. $r \circ i = id_A$ and there is a homotopy $H\colon C \times I \to C$ such that $H(x, 0) = x, H(x, 1) = r(x)$ and $H(a, t) \in A$ for all $a \in A, t \in I$. Note this implies that C is homotopy equivalent to A.

   (a) Show that $H_n(X, A) \cong H_n(X, C)$.

   (b) Show that there is a deformation retraction $C/A \to A/A$. (You may assume that the map $C \times I \to C/A \times I$ given by $(x, t) \mapsto ([x], t)$ is a quotient map.)

   (c) Show that there is an isomorphism $H_n(X/A, C/A) \to H_n(X/A, A/A) = \tilde{H}_n(X/A)$.

   (d) Show that there is a homeomorphism of pairs $(X \setminus B, C \setminus B) \to ((X \setminus B)/A, (C \setminus B)/A)$.

   (e) Excise B and B/A to complete the proof that $H_n(X, A) \cong \tilde{H}_n(X/A)$.

5. Show that $H_n(X, X \setminus \{x\})$ only depends on a closed neighborhood U of x.

6. Let $\text{Cone}(X)$ be the topological space $X \times I/(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Show that $\text{Cone}(X)$ is contractible for any X.

7. Let X be a non-empty topological space. Define the suspension $\Sigma(X)$ to be $\text{Cone}_-(X) \cup_{X \times \{1\}} \text{Cone}_+(X)$, where $\text{Cone}_-(X) = \text{Cone}_+(X) = \text{Cone}(X)$.

   (a) Use the excision and homotopy axioms to show that

   $$H_n(\Sigma(X), \text{Cone}_-(X)) \cong H_n(\text{Cone}_+(X), X).$$

   (b) Use the long exact sequence to show that $\tilde{H}_n(\Sigma(X)) \cong H_n(\Sigma X, \text{Cone}_-(X))$.

   (c) Use the long exact sequence to show that $H_n(\text{Cone}_+(X), X) \cong \tilde{H}_{n-1}(X)$.

   (d) Deduce that $H_n(\Sigma(X)) \cong \tilde{H}_{n-1}(X)$.

8. (optional) Recall the reduced chain complex of a space X is $\tilde{C}_n(X) = \begin{cases} C_n(X), \text{if } n \geqslant 0, \\ \mathbb{Z}, \text{if } n = -1. \end{cases}$ Where the map $C_0(X) \to \mathbb{Z}$ is given by $\sum_\sigma n_\sigma \sigma \mapsto \sum_\sigma n_\sigma$. Let $A \subset X$. Then define $\iota \colon \tilde{C}_n(A) \to \tilde{C}_n(X)$ as before for $n \geqslant 0$ and an isomorphism for $n = -1$.

   (a) Show that ι defines a chain map.

   (b) Define $\tilde{C}_n(X, A)$ as $\tilde{C}_n(X)/\tilde{C}_n(A)$. Define the reduced homology of a pair $\tilde{H}_n(X, A) = H_n(\tilde{C}_n(X, A)$. Show that $\tilde{H}_n(X, A) \cong H_n(X, A)$ for all n.

   (c) Deduce that there is a long exact sequence for reduced homology.

9. (optional) Let X, Y be spaces and $A \subset X$, $B \subset Y$. Show that $H_n(X \sqcup Y, A \sqcup B) \cong H_n(X, A) \oplus H_n(Y, B)$.

10. Let $x \in X$ and $y \in Y$. Let $S^0 = \{x, y\} \subset X \sqcup Y$. Show that the homomorphism $H_1(X \sqcup Y, S^0) \to H_0(S^0)$ is always 0.

4 Exercises due March 13th

1. Let X and Y be CW-complexes, describe how to get a cell structure on $X \times Y$. Hint: The cells in $X \times Y$ are products $D^n \times D^m$ for cells in D^n in X and D^m in Y.

2. Use the above to check the cell structure on the torus by giving a CW structure to S^1.

3. Show that $\Sigma(S^n)$ is S^{n+1}. The suspension Σ is defined in 3.7 above.

4. Show that the suspension $\Sigma(X)$ is homeomorphic to $X \times I/\sim$. Where $(x, t) \sim (x', t')$ if $t = t' = 0$ or $t = t' = 1$.

5. Given a map $f \colon S^n \to S^n$, define $\Sigma(f) \colon \Sigma(S^n) \to \Sigma(S^n)$ by $[(x, t)] \mapsto [(f(x), t)]$. Use the long exact sequence to show that $deg(\Sigma(f)) = deg(f)$.

6. Check the diagram chase proving that $deg(f) = \sum_i def\,|_{x_i}$.

7. Check the diagram chase showing that the entries of the matrix defining cellular homology are the degrees of the associated map. (See Hatcher pg. 141.)

8. Give the Klein bottle a cell structure and use this to compute it's homology.

9. (optional) Let v, w be vertices of CW complexes X, Y respectively. Use cellular homology to show that $H_n(X \vee Y) = H_n(X) \times H_n(Y)$. Where $X \vee Y = X \sqcup Y/v \sim w$.

5 Exercises due March 20th

1. Show that $\chi(X \times Y) = \chi(X)\chi(Y)$ for finite CW complexes X and Y.

2. Let X be a connected finite graph. Show that $H_1(X) = \mathbb{Z}^{1-\chi(X)}$.

3. (Optional) Let X and Y be connected finite graphs. Show that $H_1(X \times Y) = H_1(X) \times H_1(Y)$. Hint: you should use the fact that $H_1(X)$ is the abelianisation of the fundamental group.

4. Let X and Y be connected finite graphs. Use the previous questions to compute $H_2(X \times Y)$.

5. Let X and Y be finite CW complexes. Suppose that Z is a CW complex which is a subcomplex of both X and Y. Let $K = X \cup_Z Y$ be the union of X and Y identifying Z. Show that $\chi(K) = \chi(X) + \chi(Y) - \chi(Z)$.

6. Show that chain homotopy is an equivalence relation on chain maps.

7. Let $f\colon X \to Y$ be a continuous map. Let $i\colon Y \to \text{Cone}(Y)$ be defined by $y \mapsto (y, 1)$. Let $c\colon Y \to \text{Cone}(Y)$ be defined by $y \mapsto (y, 0)$.

   (a) Define a map $D\colon C_n(Y) \to C_{n+1}(\text{Cone}(Y))$. Such that $D\partial + \partial D = i_\Delta - c_\Delta$. Hint: use that $\text{Cone}(\Delta^n) = \Delta^{n+1}$.

   (b) Show that $i \circ f$ and $c \circ f$ are chain homotopic.

6 Exercises due March 27th

1. Show that if $f\colon G \times H \to N$ is a bi-linear map of abelian groups G, H, N, then $f(0, h) = f(g, 0) = 0$ for all $g \in G, h \in H$.

2. Check that the bi-linear map defined in class satisfies the boundary condition.

3. Complete the proof that the bi-linear map defines a map on homology $H_p(X, A) \times H_q(Y, B) \to H_{p+q}((X, A) \times (Y, B))$ by checking that it is independent of choice of representatives.

4. Read about the prism operator and reconcile this with the method of proving homotopy invariance using acyclic models.

7 Exercises due April 3rd

1. Use the Meyer-Vietoris sequence to compute the homology groups of the following spaces.

   (a) The Torus

   (b) $X \times S^1$.

   (c) (optional) $X \times S^n$, use the previous case and induction.

   (d) Two Mobius strips glued via a homeomorphism of their boundary.

   (e) The suspension of X.

   (f) The genus 2 surface.

8 Exercises due ...

1. Show that $\text{Hom}(\oplus_i A_i, G) \cong \prod_i \text{Hom}(A_i, G)$. You should use universal properties.

   Namely, the universal property of $\oplus_i A_i$ is that given homomorphisms $\phi_i\colon A_i \to G$ there is a unique homomorphism $\phi\colon \oplus_i A_i \to G$ such that $\phi \circ j_i = \phi_i$. Where j_i is the inclusion of the i-th factor.

   The universal property of $\prod_i B_i$ is that given homomorphisms $\psi_i\colon H \to B_i$. There is a unique homomorphism $\psi\colon H \to \prod_i B_i$ such that $p_i \circ \psi = \psi_i$. Where p_i is projection to the i-th factor.

   Using these you should be able to get maps each way which are inverse homomorphisms.

2. Show that $\text{Hom}(\mathbb{Z}, G) \cong G$.

3. Check that Hom defines a functor. That is check that $\bar{id} = id$ and $\overline{\phi \circ \tau} = \bar{\tau} \circ \bar{\phi}$.

4. Check that if $\phi \circ \tau = 0$, then $\overline{\phi \circ \tau} = 0$.

5. Show that when we apply $\text{Hom}(-, \mathbb{Z})$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ we do not get a short exact sequence. Specifically show that $\bar{i}$ is not surjective, where $i\colon \mathbb{Z} \to \mathbb{Z}$ is given by $i(n) = 2n$.

9 Exercises due ...

1. Check that $\overline{f + g} = \bar{f} + \bar{g}$.

2. Check the three properties of Ext. Namely,

   - $\text{Ext}(\mathbb{Z}, G) = 0$.
   - $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.
   - $\text{Ext}(A \oplus B, G) = \text{Ext}(A, G) \oplus \text{Ext}(B, G)$.

3. Show that the short exact sequence in the unversal coefficient theorem is split.

   Here are some steps:

   (a) Show that $H_n(X)$ is a quotient of $C_n(X)$ using the fact that $C_n(X) = Z_n \oplus B_{n-1}$.

   (b) Use this to get a map $p\colon \text{Hom}(H_n(X), G) \to \text{Hom}(C_n(X), G) = C^n(X; G)$.

   (c) Show that $p(\phi)$ is in the kernel of δ. Hence p defines a map $\text{Hom}(H_n(X), G) \to H^n(X; G)$. This map is the splitting.

10 Exercises due ...

1. Show that $\text{Hom}(G, -)$ is a functor from abelian groups to abelian groups. Note that unlike $\text{Hom}(-, H)$ this functor preserves the direction of arrows (it is covariant).

2. Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence and G is an abelian group. Then $0 \to \text{Hom}(G, A) \to \text{Hom}(G, B) \to \text{Hom}(G, C)$ is exact.

3. Check the properties of tensor product discussed in class. Namely:

   - $\mathbb{Z} \otimes G \cong G$
   - $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(n, m)\mathbb{Z}$
   - $G \otimes H \cong H \otimes G$
   - $(A \oplus B) \otimes H \cong (A \otimes H) \oplus (B \otimes H)$.

4. Prove that $G \otimes H$ satisfies the universal property discussed in class.

5. Show that if G is finite, then $G \otimes \mathbb{Q} = 0$.

6. Let $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ be a short exact sequence. Show that the sequence does not remain exact after applying $- \otimes \mathbb{Z}/2\mathbb{Z}$.

11 Tor Exercises

1. Show that Tor is independent of choice of resolution. You should mimic the proof for Ext.
2. Show that Tor is functorial. Once again follow the proof for Ext.
3. Show that Tor satisfies the properties discussed in class. Namely:
   - Tor(G, H) =Tor(H, G).
   - Tor$(\mathbb{Z}, G) = 0$.
   - Tor$(\mathbb{Z}/n\mathbb{Z}, H) = \{h \in H \mid nh = 0\}$.
   - Tor$(A \oplus B, H)$ =Tor$(A, H)\oplus$Tor(B, H).
4. Show that Tor$(G, \mathbb{Q}) = 0$ for all finitely generated abelian groups.
5. Compute the cohomology ring for the genus 2 surface.
6. Compute the cohomology ring for the space obtained by two tori and identifying their meridians.

12 Exercises on Poincare Duality and homology with coefficients (Optional)

1. Let M be a closed, connected, orientable 4-manifold. Suppose that $\chi(M) = 10$ and $H_1(M) = \mathbb{Z}/3\mathbb{Z}$.

   Use Poincare duality to compute all the homology and cohomology groups of M.

2. Let G be an abelian group. Let $C_n(X; G)$ be $C_n(X) \otimes G$. This gives a chain complex. Define $H_n(X; G)$ as the homology of this chain complex.

   Show that there is an exact sequence of the form $0 \to H_n(X)\otimes G \to H_n(X; G) \to$Tor$(H_{n-1}(X), G) \to 0$.

   The proof follows similar lines to the proof of the universal coefficient theorem.