Algebraic Topology Problem Sheet 7

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1. (a) Finish the classification of the subgroups of index 3 in F_2 using covering spaces.
   (b) Which of these covering spaces correspond to normal subgroups.
   (c) Write down a generating set for each subgroup.
   (d) There are 7 normal subgroups of index 4 in F_2. Find 4 of them using covering space theory. (Hint: the covering space will be the Cayley graph of a group of size 4. There are 2 such groups.)
2. Show that any map $f\colon \mathbb{R}P^2 \to S^1$ lifts to a map $\mathbb{R}P^2 \to \mathbb{R}$. (Hint: Given a continuous map f what is the image of f_*?)
3. Find a function $f\colon S^1 \to \mathbb{R}P^2$ which does not lift to S^2 and find one which does.
4. Find a covering space giving an infinite index subgroup of F_2 isomorphic to a free group on 3 elements.
5. Find a subgroup of F_3 which is isomorphic to a free group on two elements.
6. Show that S^1 has exactly one covering space of each degree. (Hint: what are the subgroups of $\mathbb{Z}$.)

   Deduce that the only covers of S^1 are homeomorphic to S^1 or $\mathbb{R}$.
7. (optional) What are the possible covers of $S^1 \times S^1$.