Algebraic Topology Problem Sheet 6

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1. Construct simply connected covering spaces of the following:

   (a) The Möbius band.

   (b) $S^2 \vee S^1$.

   (c) $S^1 \vee S^1$.

   (d) (Optional) The n-torus.

   For each of the above describe or draw the space and describe or add decorations to specify the covering map.

2. Draw a covering space $(\bar{X}, b)$ corresponding to an index 4 subgroup of $F\{a, b\}$. Compute $p_*(\pi_1(\bar{X}, b))$.

3. For each integer n, construct a degree n covering space X of the torus such that X is homomorphic to a torus. Construct a covering space Y of the torus such that Y is homeomorphic to $S^1 \times \mathbb{R}$.

   For each integer n, construct a degree n cover of the torus which is homeomorphic to a torus. Give a covering space of the torus by $S^1 \times \mathbb{R}$.

4. Show that the torus $(S^1 \times S^1)$ covers the Klein bottle. Find such a covering with degree 2.

5. (Optional) Give a covering space of the Klein bottle of degree n which is the torus for every even integer and a cover of the Klein bottle which is a Klein bottle of degree n for every integer.

6. Let X be a cover of degree k of the rose with n-petals. Show that the fundamental group of X is a free group with $k(n-1)+1$ generators. (Hint: X is a graph, how many vertices and edges are there in X.)

7. Deduce that for every $n \geqslant 2$, F_n is a finite index subgroup of F_2. Where F_n is the free group on n generators.