# Problem Sheet 5

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1. Let $S$ and $T$ be disjoint sets. Use universal properties (no presentations) to show that $F(S) * F(T) = F(S \cup T)$.

2. Let $i_G \colon G \to G * H$ be the natural homomorphism from $G$ to $G * H$. Find a map $r \colon G * H \to G$ such that $r \circ i_G = id_G$. Deduce that $i_G$ is injective.

3. Show the pushout of the following diagram is isomorphic to $\mathbb{Z}$.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$

4. Show that $\langle x, y \mid xyx = yxy \rangle$ is isomorphic to the pushout of the diagram below.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$

5. (a) The Möbius band is homotopy equivalent to a circle and has one boundary component homeomorphic to $S^1$. Thus we get an induced map $\phi \colon \mathbb{Z} \to \mathbb{Z}$ from the inclusion of the boundary. Show that this map is multiplication by 2.

   (b) Compute the fundamental group of the Möbius band with a disc attached to the boundary.

6. (a) From problem sheet 2 we know that the torus with a disc removed is homotopy equivalent to a figure 8 graph. The inclusion of this boundary induces a homomorpshims $\phi \colon \mathbb{Z} \to F(\{a, b\})$. Find the image of this homomorphism.

(b) Using the above or otherwise write down a presentation for the fundamental group of the genus 2 surface pictured below.

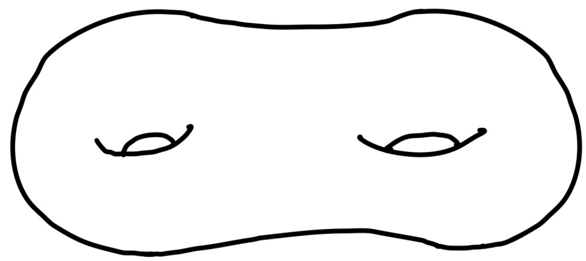

Figure 1: A genus 2 surface

7. (Optional) Assume $\phi_1 : G_0 \to G_1$ is surjective. Show that $G_1 *_{G_0} G_2$ is a quotient of $G_2$.

8. (Optional) Let $N$ be a group with maps $j_1 \colon G_1 \to N, j_2 \colon G_2 \to N$. Assume $N$ satisfies the universal property of the free product, namely, given maps $f_i \colon G_i \to G$ there is a unique map $\phi \colon N \to G$ such that $\phi \circ j_i = f_i$. Show that $N$ is isomorphic to $G_1 * G_2$.