Problem Sheet 4

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1. Let X and Y be finite sets. Show that $|X| = |Y|$ if and only if $F(X)$ is isomorphic to $F(Y)$.

2. (Optional) Remove the finiteness assumption from the previous question.

3. Let $X = \{x, y, z\}$ and $F(X)$ be the free group on X. Fix $r \in \mathbb{Z}$. Let $f\colon X \to F(X)$ be given by

$$\begin{aligned} f(x) &= x^r, \\ f(y) &= y^r, \\ f(z) &= z^r. \end{aligned}$$

   Show that the extension of f is an isomorphism.

4. Let $Y = \{a, b, c, d\}$ and X be as above. Let $g\colon Y \to F(X)$ be given by

$$\begin{aligned} g(a) &= xy, \\ g(b) &= y^2, \\ g(c) &= x^2, \\ g(d) &= yx. \end{aligned}$$

   Show that the extension does not extend to an injection.

5. Define the free abelian group on a finite set A to be the group $\mathbb{Z}^{|A|}$. There is a bijection i between A and the set of elements with a 1 in a single position.

   Show that the free abelian group satisfies the following universal property. Given an abelian group G and a map $j\colon A \to G$, there is a unique homomorphism ϕ from the free abelian group on A to G such that $j = \phi \circ i$.

6. Show that $G_1 = \langle a, b \mid aba = b\rangle$ is isomorphic to $G_2 = \langle c, d \mid c^2d^2\rangle$. Hint: in G_1 we have the equality $(abab)b^{-2} = e$

7. In this problem we will show that $D_n \cong \langle x, y \mid x^2y^n, xyxy\rangle$.

   Let $G = \langle x, y \mid x^2, y^n, xyxy\rangle$.

(a) Let r be a rotation of the n-gon through $\frac{2\pi}{n}$ and s be a reflection through an axis. Show that the function $x \mapsto s, y \mapsto r$ gives a homomorphism $G \to D_n$.

(b) Deduce that this homomorphism is surjective.

(c) Using the relation $xyxy$ to show that every element of G can be written in the form $x^k y^l$ for some $k < 2$, $l < n$.

(d) Deduce that $|G| \leqslant 2n$.

(e) Deduce that the function defined in a) is an isomorphism.

8. Let $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_2 = \langle X_2 \mid R_2 \rangle$. Give a presentation (with proof) for $G_1 \times G_2$.

9. (Optional) Assume $\phi_1 : G_0 \to G_1$ is surjective. Show that $G_1 *_{G_0} G_2$ is a quotient of G_2.