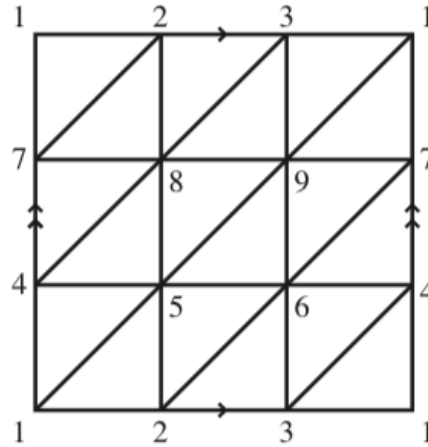


Algebraic Topology Sheet 3

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October 9, 2019

1. Show that $\pi_1(X \times Y, (x, y)) = \pi_1(X, x) \times \pi_1(Y, y)$. (Hint: use the projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$.)
2. (a) Consider the following triangulation of the torus.
 (b) Deduce that the fundamental group of the torus is \mathbb{Z}^2 .



Let x and y be the loops $(1, 2, 3, 1)$ and $(1, 4, 7, 1)$, and let K be the union of these two loops (ie. K comes from the boundary of the square).

- (a) Show that any edge path that starts and ends on K but with the remainder of the path missing K is equivalent to an edge path lying entirely in K .
- (b) Show that any edge loop based at 1 is equivalent to an edge loop lying entirely in K .

- (c) Deduce that any edge loop based at 1 is equivalent to a word in the alphabet $\{x, y\}$.
 - (d) Show that the edge loops xy and yx are equivalent.
 - (e) Deduce that any edge loop based at 1 is equivalent to $x^m y^n$, for $m, n \in \mathbb{Z}$.
 - (f) Prove that if $x^m y^n \sim x^M y^N$, then $m = M$ and $n = N$. (Hint: define 2 winding numbers as in the proof that $\pi_1(S^1) = \mathbb{Z}$, one for the loop x and one for the loop y .)
 - (g) Deduce that the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.
3. A *retraction* of X onto $A \subset X$ is a map $r: X \rightarrow A$, such that $r \circ i: A \rightarrow A$ is the identity map.
- (a) Show that there is no retraction D^2 onto S^1 . (Hint: Consider the induced maps on fundamental groups.)
 - (b) Show that every map $f: D^2 \rightarrow D^2$ has a fixed point. To start you off, suppose that $f(x) \neq x$ for all $x \in D^2$. Use the pairs $(x, f(x))$ to construct a retraction D^2 onto S^1 , reaching a contradiction. Thus, f must have a fixed point.
 - (c) Use similar reasoning to show that any map $f: D^2 \rightarrow D^2$ which is the identity on the boundary must be surjective.
4. Show the following:
- (a) \mathbb{R} is not homeomorphic to \mathbb{R}^n , $n \geq 2$.
 - (b) \mathbb{R}^2 is not homeomorphic to \mathbb{R}^n , $n > 2$.
- Note the following:
- (a) The fundamental group is invariant under homeomorphism.
 - (b) Being path connected is invariant under homeomorphism.
5. (Optional) Show that the Mobius strip M does not have a retraction onto its boundary circle.