

Algebraic Topology: Problem Sheet 2

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1. Let α be the antipodal map on S^n ($\alpha(x) = -x$). Show that if n is odd, then $\alpha \simeq id_{S^n}$.
2. Let $f, g : X \rightarrow S^n$ be maps satisfying $f(x) + g(x) \neq 0$ for all $x \in X$. Prove $f \simeq g$. Hint: Consider how to do this in \mathbb{R}^{n+1} and normalise.
3. Recall that a space is path connected if for every $x, y \in X$ there is a map $\gamma : I \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.
Show that any contractible space is path connected.
4. Let X be a contractible space. Show that $X \times Y \simeq_{h.e.} Y$.
5. Let X be a path connected space. Show that the following statements are equivalent:
 - (a) X is contractible.
 - (b) For any space Y , any 2 maps $f, g : Y \rightarrow X$ are homotopic.
 - (c) For any path connected space Y , any 2 maps $k, l : X \rightarrow Y$ are homotopic.
6. Show that $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} .
7. Consider each capital letter as a topological space. Divide them into their homotopy equivalence classes. Hint: There are 3 classes. The letters are reproduced below for posterity.
ABCDEFGHIJKLMNOPQRSTUVWXYZ
8. Show the following 3 spaces are homotopy equivalent.

- (a) The union of 2 circles with one point identified (A figure 8 graph).
 - (b) A torus with a disk removed (cf. Problem sheet 1)
 - (c) $\mathbb{R}^2 \setminus \{(-1, 0), (1, 0)\}$.
9. Let $f, g : S^m \rightarrow S^n$ be two maps. Prove that if $m < n$, then $f \simeq g$. (Hint: Use the simplicial approximation theorem.)
 10. (Optional) The wedge product of two non-empty spaces $X \vee Y$ is quotient of the disjoint union identifying one point in X with one point in Y . Show that $\mathbb{R}^2 \setminus \{p_1, \dots, p_k\}$ is homotopy equivalent to $S^1 \vee \dots \vee S^1$ where there are k copies of S^1 .
 11. (optional) Show that $\mathbb{R}^n \setminus \{p_1, \dots, p_k\}$ is homotopy equivalent to $S^{n-1} \vee \dots \vee S^{n-1}$ where there are k copies of S^{n-1} .
 12. (Doubly Optional) Let $f, g : S^{n-1} \rightarrow X$ be maps such that $f \simeq g$. Prove $X \cup_f D^n \simeq_{h.e.} X \cup_g D^n$.