Algebraic Topology Midterm

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This exam is due on Friday 18th by 6pm. You are allowed to use the lecture notes by Marc Lackenby and any solutions to homeworks. You should not use the internet, and should not discuss with fellow students. Please sign your final exam to state that you have adhered to these guidelines.

You may assume that $\pi_1(S^1, 1) = \mathbb{Z}$. You can also use the fact that $\mathbb{Z} \cong F(\{a\})$.

1. Recall that $f\colon X \to Y$ is a homotopy equivalence if there is a map $g\colon Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

   (a) Let $f\colon X \to Y$ be a continuous map. Show that f is a homotopy equivalence if and only if there exists maps $g, h\colon Y \to X$ such that both $g \circ f \simeq id_X$ and $f \circ h \simeq id_Y$. (Hint: Consider the map $g \circ f \circ h$.)

   (b) Let f be a homotopy equivalence. Suppose $f \simeq k$, show that k is a homotopy equivalence.

2. Let $\mathrm{Hom}(G, N)$ denote the set of group homomorphims from G to N. Given a homomorphism $f\colon G \to H$ there is a map $\bar{f}\colon \mathrm{Hom}(H, N) \to \mathrm{Hom}(G, N)$ given by $\bar{f}(\phi) = \phi \circ f$.

   (a) Suppose that $f\colon G \to H$ is a surjection. Show that if $\phi \circ f = \psi \circ f$, then $\phi = \psi$ for all homomorphisms $\phi, \psi\colon H \to N$.

   (b) Deduce that in the case that f is a surjection, then $\bar{f}$ is an injection.

   (c) Let $F(X)$ be the free group on a finite set X.
   Show that $|\mathrm{Hom}(F(X), \mathbb{Z}/3\mathbb{Z})| = 3^{|X|}$. Hint: Use the universal property.

   (d) Deduce that if X and Y are finite sets and $|X| < |Y|$, then there is no surjection $F(X) \to F(Y)$.

3. Let $Y \subset X$ be topological spaces. Let $i\colon Y \to X$ be the inclusion map. Assume there is a retraction $r\colon X \to Y$ (Recall a retraction is a map r such that $r \circ i = id_Y$). Let $b \in Y$ be a basepoint.

   (a) Show that $r_*\colon \pi_1(X, b) \to \pi_1(Y, b)$ is surjective.

   (b) Now suppose that X is contractible. Show that Y is also contractible. Hint: Use the criterion from problem sheet 2 and show that $f \circ r \circ i = f$ for any function $f\colon Y \to Z$.

4. Consider the topological space pictured below, formed from a triangulated rectangle by identifying the two vertices labelled "1". (Imagine identifying two corners of this exam.)

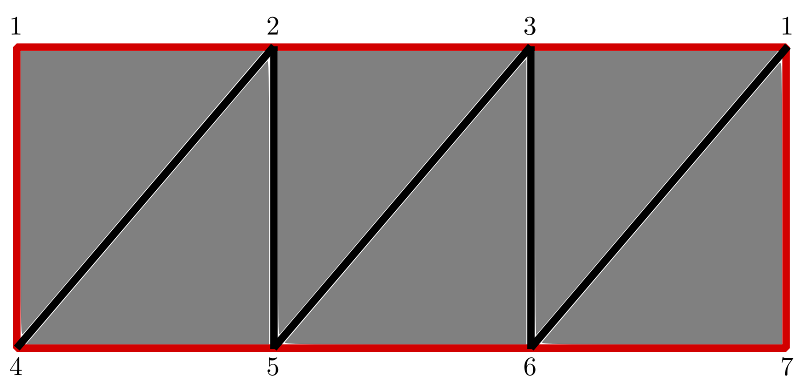
   (a) Write down the simplicial complex obtained from this diagram.

   (b) Show that the space is homotopy equivalent to S^1.

   (c) Deduce that the fundamental group of X is $\mathbb{Z}$.

   (d) Consider the subcomplex highlighted in red. Explicitly describe the fundamental group of this subcomplex.

   (e) Show that there is no retraction of $|K|$ onto this subcomplex.



5. Let $\{(x, 0, 0) \mid x \in \mathbb{R}\} = \mathbb{R} \subset \mathbb{R}^3$. Let $S^1 = \{(x, y, 0, 0) \mid x^2 + y^2 = 1\} \subset S^3$.

   (a) Show that the map $q\colon \mathbb{R}^3 \setminus \mathbb{R} \to \mathbb{R}^2 \setminus \{(0, 0)\}$ given by $q(x, y, z) = (y, z)$ is a homotopy equivalence by giving the inverse map and the required homotopies

   (b) Compute the fundamental group of $\mathbb{R}^3 \setminus \mathbb{R}^1$.

   (c) Compute the fundamental group of $S^3 \setminus S^1$.