Algebraic Topology Final Exam 2018

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December 13, 2019

This exam is due on Monday 16th December at noon. You are allowed to use the lecture notes by Marc Lackenby and resources linked on the course webpage. You should not use the internet, and should not discuss with fellow students. Please sign your final exam to state that you have adhered to these guidelines.

You should hand this in via email (preferable), directly to me (Office SEC LL010), my mailbox in the department office or the homework mailbox. If you choose the latter please email to inform me you have handed it in.

Throughout you may assume that fundamental group S^1 is $\mathbb{Z}$, the fundamental group of $S^1 \times S^1$ is $\mathbb{Z}^2$ and the fundamental group of $\mathbb{R}P^2$ is $\mathbb{Z}/2\mathbb{Z}$.

1. Let $F(X)$ be the free group on the set X. For $g \in F(X)$ let $|g|$ be the number of letters in the reduced representative for g.

   (a) (3 marks) Show that $|g| = 0$ if and only if $g = e$.

   (b) (6 marks) Show that $|g^n| > |g|$ if $g \neq e$.

   (c) (2 marks) Deduce that every non-trivial element on $F(X)$ has infinite order.

   (d) (2 marks) Show that every non-trivial element of $\mathbb{Z}^2$ has infinite order.
   Let $G = \langle x, y, t \mid xyx^{-1}y^{-1}, txt^{-1} = y, tyt^{-1} = x, t^2 \rangle$.

   (e) (2 marks) Describe a cell complex K with fundamental group G. (You do not have to prove that it has the correct fundamental group.)

   (f) (2 marks) Show that $x \mapsto (1\,2\,3), y \mapsto (1\,3\,2), t \mapsto (1\,2)$ defines a homomorphism $G \to S_3$.

   (g) (4 marks) Show that the order of t is 2.

   (h) (4 marks) Show that K is not homotopy equivalent to any graph. Show that K is not homotopy equivalent to a torus.

2. Let X be a topological space. We say that X has *the fixed point property* if for any map $f\colon X \to X$ there exists $x \in X$ such that $f(x) = x$.

   (a) (6 marks) Let $r\colon X \to A$ be a retraction. Show that if X has the fixed point property, then so does A.

   (b) (3 marks) Show that $\mathbb{R}^2$ does not have the fixed point property.

   (c) (3 marks) Show that D^2 has the fixed point property.

   (d) (3 marks) Show that there is no retraction D^2 to $\{(x, y) \mid x^2 + y^2 < 1\}$.

3. Let $M_1 = M_2 = S^1 \times D^2$. Let $T = S^1 \times S^1$.

   (a) (2 marks) Show that M_1 is homotopy equivalent to S^1.

   (b) (1 mark) Deduce that $\pi_1(M_1, b) = \mathbb{Z}$.

   (c) (6 marks) Let $f_1\colon T \to M_1$ be given by $f_1((x, y)) = (x, y)$.
   Let $f_2\colon T \to M_1$ be given by $f_2((x, y)) = (y, x)$.
   Let $g\colon T \to M_2$ be given by $g((x, y)) = (x, y)$.
   Compute f_{1*}, f_{2*}, g_*.

   (d) (6 marks) Let $X_i = M_1 \cup_{S^1 \times S^1} M_2/\sim$, the equivalence relation begin given by $f_i((x, y)) = g((x, y))$. Compute the fundamental groups $\pi_1(X_1, b)$ and $\pi_1(X_2, b)$.

4. Let T be a torus. Let $f\colon \mathbb{R}P^2 \to T$ be a continous function. Recall $\mathbb{R}^2 \to T$ given by $(x, y) \mapsto (e^{2\pi x}, e^{2\pi y})$ is a covering map.

   (a) (2 marks) Define what it means for f to *lift* to $\mathbb{R}^2$.

   (b) (2 marks) Give a sufficient criterion for f to lift.

   (c) (3 marks) Show that f always lifts to $\mathbb{R}^2$.

   (d) (1 mark) Show that any two maps $g, h\colon \mathbb{R}P^2 \to \mathbb{R}^2$ are homotopic.

   (e) (2 marks) Show that f is homotopic to a constant function.

5. (a) (2 marks) State the simplicial approximation theorem.

   (b) (8 marks) Let $\mathcal{G}$ be a graph show that any two maps from $\mathcal{G} \to S^2$ are homotopic.

   (c) (5 marks) Find two maps from $(S^1 \vee S^1, b) \to (S^1 \times S^1, (1, 1))$ that are not homotopic relative to b.