Solutions for problem sheets

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Sheet 1

1. See picture

2. The n-simplex has a simplicial structure given by $(V, \mathcal{P}(V))$ where $|V| = n + 1$.

3. Define the maps from $(V, \Sigma) \to (V, \mathcal{P}(V))$ by $v \mapsto v$. This is a simplicial map since every simplex exists in $\mathcal{P}(V)$.

4. Let (V, Σ) be a simplicial complex.

   If we had a smaller triangulation then it would have < 3 vertices. We can now enumerate the possible complexes with < 3 vertices. These are a point, S^0 and an interval.

5. The dimension of K is $\max\{|\sigma| - 1 \mid \sigma \in \Sigma\} = d$. If $\sigma \in \Sigma$ then $\mathcal{P}(\sigma) \subset \Sigma$ so $n = |\Sigma| \geq |\mathcal{P}(\sigma)| = 2^{d+1}$. Thus $d \leq \log_2(n) - 1$.

6. See pictures. To deduce the cell structures we just count the number of vertices and edges in each picture after identification. Each picture has a single 2-cell.

Sheet 2

1. Recall

$$\begin{aligned} S^{2n-1} &= \{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \mid \Sigma x_i^2 = 1\} \\ &= \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \Sigma |z_i|^2 = 1\}. \end{aligned}$$

We can thus set up a homotopy $H : S^{2n-1} \times I \to S^{2n-1}$ by

$$H((z_1, \dots, z_n), t) = (e^{i\pi t} z_1, \dots, e^{i\pi t} z_n).$$

We can see that at $t = 0$ this is the identity map and at $t = 1$ this is the antipodal map.

2. Set up a homotopy $H : S^n \times I \to S^n$ as follows.

$$H(x, t) = \frac{tf(x) + (1-t)g(x)}{\|tf(x) + (1-t)g(x)\|}.$$

We must check that this is well defined. This is the same as showing that $\|tf(x) + (1-t)g(x)\| \neq 0$.

$$\|tf(x) + (1-t)g(x)\| = 0$$
$$\Rightarrow tf(x) + (1-t)g(x) = 0$$
$$\Rightarrow tf(x) = (t-1)g(x)$$

By taking norms of both sides, we see that $t = \frac{1}{2}$

$$\Rightarrow f(x) = -g(x)$$

Thus this homotopy is well defined. And the maps f, g are homotopic.

3. $a) \implies b)$

Let b be a point of X and $H : X \times I \to X$ be a homotopy from the identity map on X to the constant map c_b.

Compose with the map id_X and see the following equivalences using H

$$f = \mathrm{id}_X \circ f \simeq c_b \circ f = c_b \simeq c_b \circ g \simeq \mathrm{id}_X \circ g = g$$

$b) \implies a)$

Take $\mathrm{id}_X \colon X \to X$ and $c_b \colon X \to X$. These are homotopic by assumption, thus X is contractible.

$a) \implies c)$

Once again compose with the identity map and use the homotopy H from above. Also let γ be a path from $f(b)$ to $g(b)$ and note that this defines a homotopy from $c_{f(b)}$ to $c_{g(b)}$. We can now get the following chain of homotopies.

$$f = f \circ \mathrm{id}_X \simeq f \circ c_b = c_{f(b)} \simeq c_{g(b)} = g \circ c_b \simeq g \circ \mathrm{id}_X = g.$$

c) $\implies$ *a)*

Once again just take $\mathrm{id}_X\colon \to X$ and $c_b\colon X \to X$ and these are homotopic by assumption.

4. Use the map $x \mapsto \frac{x}{|x|}$ and the straight line homotopy in $\mathbb{R}^n$.

5. The equivalence classes are $\{A, D, O, P, Q, R\}, \{B\}$,

   $\{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z\}$ This font might be incorrect but for the font on the sheet these are the correct equivalence classes.

6. Add pictures

7. Let $f, g : S^m \to S^n$ be two maps and let $m < n$. Both spaces are homeomorphic to simplicial complexes as follows. Let V_i be a set with $i + 1$ elements. ^{i} is homeomorphic to $(V_i, \mathcal{P}(V_i) \setminus \{V_i\})$. This is the boundary of an $i - simplex$.

   We can now apply the simplicial approximation theorem to the map f to get a simplicial map h from a subdivision of the simplicial complex. Noting that the triangulation of S^m has simplices of dimension at most m we see that the map $|h|$ is not surjective.

   Thus the image is contained in $S^n \setminus \{*\}$. This is homeomorphic to $\mathbb{R}^n$ and by the previous question is thus homotopic to a constant map. Similar reasoning for g shows that the maps are both homotopic to constant maps and as S^n is path connected these maps are homotopic.

8. Add later

Sheet 3

1. Let $p : X \times Y \to X$ be the map $p(x, y) = x$ and $q : X \times Y \to Y$ be the map $q(x, y) = y$. These are continuous maps, so we get maps $p_* : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x)$ and $q_* : \pi_1(X \times Y, (x, y)) \to \pi_1(Y, y)$.

Define a map $\phi : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$ $\phi(g) = (p_*(g), q_*(g))$. We must show that this map is surjective and injective.

Given an element $(h, k) \in pi_1(X, x) \times \pi_1(Y, y)$. Let $\alpha : I \to X$ be a loop representing h and $\beta : I \to Y$ be a loop representing k. The loop $\gamma(t) = (\alpha(t), \beta(t))$ is a loop in $X \times Y$. It is easy to see that $\phi([\gamma]) = (h, k)$. Thus ϕ is surjective.

To show that it is injective pick an element γ of the kernel of ϕ. Since it is in the kernel $\phi(\gamma) = (p_*(\gamma), q_*(\gamma)) = (e, e)$. Let H be a homotopy showing that $p_*(\gamma)$ is trivial and K be a homotopy showing that $q_*(\gamma)$ is trivial. The map $L : X \times Y \times I \to X \times Y$, defined by $L(a, b, t) = (H(a, t), K(b, t))$. We can see that this is a homotopy between the constant map and γ. Thus ϕ is injective.

2. (a) Since the image of the interior of any 2-simplex is missed we can push out from this point to the boundary. This gives a homotopy from the original loop to one that is solely contained in K.
   (b) This follows directly from part a).
   (c) Since we can remove pauses and backtracks we see that if the loop begins (1, 2) it must continue to 3 and thus we see an x. This reasoning can be applied to the other possible loops starting at 1. Thus we get a word in x and y.
   (d) See Pictures
   (e) We can move all the x's to the left since x and y commute thus we end with a loop of the form $x^n y^m$.
   (f) Define two winding numbers. The first LR counting left-right movement given by:

$$\begin{aligned} &\#\{\text{occurence of } (2,3),(2,6),(5,6),(5,9),(8,9),(8,3)\} \\ -&\#\{\text{occurences of } (3,2),(6,2),(6,5),(9,5),(9,8),(8,3)\} \end{aligned}$$

   We can see that this is invariant under elementary expansions and contractions and $LR(x^m y^n) = m$. Thus we see that $m = M$. Defining a similar winding number for up-dowm movement we arrive at the conclusion.
   (g) We can now set up an isomorphism with $\mathbb{Z}^2$ by $x^m y^n \mapsto (m, n)$.

3. (a) A retraction $r : D^2 \to S^1$ is a map such that $r \circ i : S^1 \to S^1$ is the identity map. We can now look at the induced maps on fundamental groups noting that $\pi_!(S^1, 1) = \mathbb{Z}$ and $\pi_1(D^2, 1) = \{e\}$. We have the following maps.

$$r_* : \mathbb{Z} \to \{e\}$$

$$i_* : \{e\} \to \mathbb{Z}$$

such that $r_* \circ i_*$ is the identity on $\mathbb{Z}$ however since it factors through $\{e\}$ it is the trivial map.

(b) Assume that $f : D^2 \to D^2$ is a map with no fixed points. Construct a map $g : D^2 \to S^1$ as follows. Since $f(x) \neq x$ we can find a real number $s > 0$ such that $\|(1-s)f(x) + sx\|$. Define $g(x)$ to be this point. This is a continuous map which is the identity on the circle. By part a) we see that this map cannot exist and so f must have had a fixed point.

(c) Assume that we had such a map f which was not surjective. Suppose that p is a point not in the image. For each point of $D^2 \setminus \{p\}$ take the line from p through x and define $r(x)$ as the point on S^1 that intersects this line. Then the composition $r \circ f$ defines a retraction onto S^1. This contradicts a).

4. If any of these spaces were homeomorphic, then they would also be homeomorphic after removing a point from each. We know that $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to S^n. And by the previous sheet S^n is simply connected for $n > 1$. Thus the results follow.

5. A retraction $r : M \to S^1$ is a map such that $r \circ i : S^1 \to S^1$ is the identity map. We can now look at the induced maps on fundamental groups noting that $\pi_!(S^1, 1) = \mathbb{Z} = \pi_1(M, 1)$. We have the following maps.

$$r_* : \mathbb{Z} \to \mathbb{Z}$$

$$i_* : \mathbb{Z} \to \mathbb{Z}$$

such that $r_* \circ i_*$ is the identity on $\mathbb{Z}$ however the map i_* is multiplication by 2, this map does not have an inverse.

Sheet 4

1. Firstly, suppose that $|X| = |Y|$, we will show that $F(X) \cong F(Y)$. Let $f\colon X \to Y$ be a bijection. By the universal property of free groups we get homomorphisms $\phi\colon F(X) \to F(Y)$ and $\psi\colon F(Y) \to F(X)$ such that $\phi|_X = f$ and $\psi|_Y = f^{-1}$. We will show that $\phi \circ \psi = id_{F(Y)}$ and $\psi \circ \phi = id_{F(X)}$.

   Look at $\psi \circ \phi(x)$ for $x \in X$. By the choice of ϕ we see that $\psi \circ \phi(x) = \psi(f(x))$ and by choice of ψ we have that $\psi(f(x)) = f^{-1}(f(x)) = x$. Thus $\psi \circ \phi$ is the identity on X and hence gives the identity on $F(X)$. The proof for the other composition is similar.

   For the other direction, suppose that $F(X) \cong F(Y)$, we will show that $|X| = |Y|$. The universal property of free groups tells us that $Hom(F(X), G) = Map(X, G)$. Thus if $F(X) \cong F(Y)$, we have that $Map(X, G) = Map(Y, G)$ for all groups G. If G is a finite group then $|Map(X, G)| = |G|^{|X|}$. Taking logs of both sides we see that $|X| = |Y|$.

2. Optional

3. Let w be a reduced word in x, y, z. Then $f(w)$ is also a reduced word. Thus if w is non-trivial we can see that $f(w)$ is non-trivial. Thus f is an injection and an isomorphism onto its image.

4. To see that the extension of f is not an injection, note that $(xy)(y^{-2})(yx)(x^{-2}) = e$. Thus under the extension $ab^{-1}cd^{-1}$ is a reduced word that maps to the identity.

5. Let G be an Abelian group. Let $A = \{a_1, \dots, a_n\}$. Define $\phi\colon \mathbb{Z}^n \to G$ by $\phi((l_1, \dots, l_n)) = l_1 f(a_1) + \cdots + l_n f(a_n)$. It is easy to check that this defines a homomorphism. It also satisfies the property that $\phi(j(a)) = f(a)$.

   To see uniqueness note that $\mathbb{Z}^n$ is generated by $j(A)$ and so if g is any other homomorphism such that $g \circ j = f$, then g and ϕ agree on a generating set of $\mathbb{Z}^n$ and so agree on the whole group.

6. We set up an isomorphism by

$$\begin{aligned} c &\mapsto ab \\ d &\mapsto b^{-1} \end{aligned}$$

The hint tells us that this will define a homomorphism. The map defined by

$$a \mapsto cd,$$
$$b \mapsto d^{-1},$$

defines an inverse homomorphism.

7. (a) To see that this is homomorphism check that all the relations hold. For x^2, y^n this is clear. For $xyxy$ we need to check pictorially but this relation also holds.

   (b) Since r, s generate D_n we see that this homomorphism is surjective.

   (c) The relation $xyxy$ can be rewritten as $xy = y^{-1}x^{-1} = y^{-1}x$. Repeatedly using this relation we can move all the y's to the end of the word. Thus we see that each element can be represented in the form $x^k y^l$. Also using the relations x^2 and y^n we can assume that $k < 2$ and $l < n$.

   (d) The previous part shows that each element is of the form $x^k y^l$ with 2 choices for k and n choices for l. We deduce that there are at most $2n$ possible elements of G.

   (e) Since the group G is finite and has at least $2n$ elements by b) and at most $2n$ elements by d) we see that the homomorphism defined is in fact an isomorphism.

8. A presentation for the product $G_1 \times G_2$ is

$$\mathcal{Q} = \langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup \{xyx^{-1}y^{-1} : x \in X_1, y \in X_2\}\rangle$$

   To prove that this gives the required presentation we need an isomorphism $\phi\colon \mathcal{Q} \to G_1 \times G_2$.

   There is a map $f_1\colon X_1 \sqcup X_2 \to G_1$ given by

$$f_1(x) = x, \forall x \in X_1$$

$$f_1(y) = e, \forall y \in X_2$$

Define $f_2\colon X_1 \sqcup X_2 \to G_2$ similarly. These extend to homomorphisms $\phi_i\colon \mathcal{Q} \to G_i$ by van Dyk's lemma. This gives us the map $\phi\colon \mathcal{Q} \to G_1 \times G_2$ by

$$\phi(g) = (\phi_1(g), \phi_2(g)).$$

We must check this is surjective and injective.

To prove that it is surjective it is enough to note that it maps onto the set $X_1 \times \{e\} \cup \{e\} \times X_2$, this is clear from the definition of ϕ_i.

To check that it is injective take a word w in the kernel. Using the added relations we can assume that any occurence of a letter of X_1 appears to the left of any letter of X_2. Since in the group $\mathcal{Q}$

$$yx = yx(x^{-1}y^{-1}xy) = xy.$$

Thus we can assume w has the form w_1w_2 where w_i is a word in X_i. Thus the image of this word under ϕ is (w_1, w_2). Since w is in the kernel this must be trivia, i.e. $w_i \in \langle\langle R_i \rangle\rangle$. And so we see that $w_1w_2 \in \langle\langle R_1 \cup R_2 \rangle\rangle$ and the map is injective.

Sheet5

1. Let $i\colon S \cup T \to F(S \cup T)$ be the inclusion. Define $j\colon S \cup T \to F(S) * F(T)$ be the function sending $s \in S$ to $s \in F(S)$ and to $F(S) * F(T)$ by the natural map $F(S) \to F(S) * F(T)$. Define $j|_T$ similarly.

   By the universal property of free groups we get a homomorphism $\phi\colon F(S \cup T) \to F(S) * F(T)$. Also the function $S \to S \cup T \to F(S \cup T)$ gives us a homomorphism $F(S) \to F(S \cup T)$, similarly for T. By the universal property of the free product we get a homomorphism $\psi\colon F(S) * F(T) \to F(S \cup T)$.

   We will show that both compositions are the identity. Consider $\psi \circ \phi\colon F(S \cup T) \to F(S \cup T)$. Let $s \in S$, then $\psi \circ \phi(s) = \psi(j(s)) = s \in F(S \cup T)$. A similar argument for T shows that $\psi \circ \phi = id_{F(S \cup T)}$.

   Now look at $\phi \circ \psi\colon F(S) * F(T) \to F(S) * F(T)$. Since the restriction to the image of $F(S)$ sends an element of S to itself thus restricted to $F(S)$ we have the identity map. Similarly for $F(T)$. Thus we can see that $\phi \circ \psi$ is the identity on $F(S) * F(T)$.

2. Define homomorphisms $\phi\colon G \to G$ by $\phi(g) = g$ and $\psi\colon H \to G$ by $\psi(h) = e$. By the universal property of the pushout we get a homomorphism $r\colon G * H \to G$ such that $r \circ i_G = id_G$. Thus we see that i_G is injective.

3. The pushout has presentation

$$\langle x, y \mid y = x^2 \rangle.$$

Using a Tietze transformation we see that this is isomorphic to the group $\langle y \mid \rangle = \mathbb{Z}$.

4. We know that the pushout has the presentation

$$\langle a, b \mid a^2 = b^3 \rangle.$$

We can set up a map to the required group by

$$\begin{aligned} a &\mapsto xyx, \\ b &\mapsto xy. \end{aligned}$$

We must check that a^2b^{-3} maps to the identity. This is the same as showing that $(xyx)(xyx) = xyxyxy$. By the relation we can replace the second xyx with yxy and we are done.

We can see that $b^{-1}a$ maps to x and $a^{-1}b^2$ maps to y. Thus we can create an inverse map

$$\begin{aligned} x &\mapsto b^{-1}a, \\ y &\mapsto a^{-1}b^2. \end{aligned}$$

We need to check this defines a homomorphism, this is the same as saying that the image of xyx equals the image of yxy. Which is the same as the following

$$\begin{aligned} b^{-1}aa^{-1}b^2b^{-1}a &= a^{-1}b^2b^{-1}aa^{-1}b^2 \\ a &= b^3a^{-1} \end{aligned}$$

This last line is true since $a^2 = b^3$.

This map is inverse to the previous map and as such they are isomorphisms.

5. See attached pictures.
6. See attached pictures.