Problem Sheet 4

Robert Kropholler

March 16, 2018

1. Let X and Y be finite sets. Show that $|X| = |Y|$ if and only if $F(X)$ is isomorphic to $F(Y)$.

2. (Optional) Remove the finiteness assumption from the previous question.

3. Let $X = \{x, y, z\}$ and $F(X)$ be the free group on X. Fix $r \in \mathbb{Z}$. Let $f\colon X \to F(X)$ be given by

$$\begin{aligned} f(x) &= x^r, \\ f(y) &= y^r, \\ f(z) &= z^r. \end{aligned}$$

   Show that the extension of f is an isomorphism.

4. Let $Y = \{a, b, c, d\}$ and X be as above. Let $g\colon Y \to F(X)$ be given by

$$\begin{aligned} g(a) &= xy, \\ g(b) &= y^2, \\ g(c) &= x^2, \\ g(d) &= yx. \end{aligned}$$

   Show that the extension does not extend to an injection.

5. Let S and T be disjoint sets. Use universal properties (no presentations) to show that $F(S) * F(T) = F(S \cup T)$.

6. Define the free abelian group on a finite set A to be the group $\mathbb{Z}^{|A|}$. There is a bijection i between A and the set of elements with a 1 in a single position.

   Show that the free abelian group satisfies the following universal property. Given an abelian group G and a map $j\colon A \to G$, there is a unique homomorphism ϕ from the free abelian group on A to G such that $j = \phi \circ i$.

7. Let $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_2 = \langle X_2 \mid R_2 \rangle$. Give a presentation (with proof) for $G_1 \times G_2$.

8. Show that $G_1 = \langle a, b \mid aba = b \rangle$ is isomorphic to $G_2 = \langle c, d \mid c^2d^2 \rangle$. Hint: in G_1 we have the equality $(abab)b^{-2} = e$

9. Let X, Y be spaces. Find a space with fundamental group $\pi_1(X, x) * \pi_1(Y, y)$.

10. Let $i_G \colon G \to G * H$ be the natural homomorphism from G to $G * H$. Find a map $r \colon G * H \to G$ such that $r \circ i_G = id_G$. Deduce that i_G is injective.

11. Show the pushout of the following diagram is isomorphic to $\mathbb{Z}$.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$

12. Show that $\langle x, y \mid xyx = yxy \rangle$ is isomorphic to the pushout of the diagram below.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$

13. (Optional) Assume $\phi_1 : G_0 \to G_1$ is surjective. Show that $G_1 *_{G_0} G_2$ is a quotient of G_2.

14. (Optional) Let N be a group with maps $j_1 \colon G_1 \to N, j_2 \colon G_2 \to N$. Assume N satisfies the universal property of the free product, namely, given maps $f_i \colon G_i \to G$ there is a unique map $\phi \colon N \to G$ such that $\phi \circ j_i = f_i$. Show that N is isomorphic to $G_1 * G_2$.