Algebraic Topology Midterm

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March 9, 2018

This exam is due on Monday 12th at noon. You are allowed to use the lecture notes by Marc Lackenby. You should not use the internet, and should not discuss with fellow students. Please sign your final exam to state that you have adhered to these guidelines.

You may assume that $\pi_1(S^1, 1) = \mathbb{Z}$ and $S^n \smallsetminus \{p\}$ is homeomorphic to $\mathbb{R}^n$.

1. Let $\{(x, y, 0, 0) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2 \subset \mathbb{R}^4$. Let $\{(x, 0, 0) \mid x \in \mathbb{R}\} = \mathbb{R} \subset \mathbb{R}^3$. Let $S^2 = \{(x, y, z, 0, 0) \mid x^2 + y^2 + z^2 = 1\} \subset S^4$.

   (a) Show that the map $p\colon \mathbb{R}^4 \smallsetminus \mathbb{R}^2 \to \mathbb{R}^3 \smallsetminus \mathbb{R}$ given by $p(x, y, z, t) = (x, z, t)$ is a homotopy equivalence.

   (b) Show that the map $q\colon \mathbb{R}^3 \smallsetminus \mathbb{R} \to \mathbb{R}^2 \smallsetminus \{(0, 0)\}$ given by $q(x, y, z) = (y, z)$ is a homotopy equivalence.

   (c) Compute the fundamental group of $\mathbb{R}^4 \smallsetminus \mathbb{R}^2$.

   (d) Compute the fundamental group of $S^4 \smallsetminus S^2$.

2. Let $f\colon X \to Y$ be a continuous map. Show that f is a homotopy equivalence if and only if there exists maps $g, h\colon Y \to X$ such that both $g \circ f \simeq id_X$ and $f \circ h \simeq id_Y$. (Hint: Consider the map $g \circ f \circ h$.)

3. Let $\mathrm{Hom}(G, N)$ denote the set of group homomorphims from G to N. Given a homomorphism $f\colon G \to H$ there is a map $\bar{f}\colon \mathrm{Hom}(H, N) \to \mathrm{Hom}(G, N)$ given by $\bar{f}(\phi) = \phi \circ f$.

   (a) Show that if $f\colon G \to H$ is a surjection, then $\bar{f}$ is an injection.

   (b) Let $F(X)$ be the free group on a set X.
   Show that $|\mathrm{Hom}(F(X), \mathbb{Z}/2\mathbb{Z})| = |\mathcal{P}(X)|$. Here $\mathcal{P}(X)$ is the power set of X.

   (c) Deduce that if X and Y are finite sets and $|X| < |Y|$, then there is no surjection $F(X) \to F(Y)$.

4. Let $Y \subset X$ be topological spaces. Let $i\colon Y \to X$ be the inclusion map. Assume there is a retraction $r\colon X \to Y$ (Recall a retraction is a map r such that $r \circ i = id_Y$). Let $b \in Y$ be a basepoint.

   (a) Show that $r_*\colon \pi_1(X, b) \to \pi_1(Y, b)$ is surjective.

   (b) Show that $i_*\colon \pi_1(Y, b) \to \pi_1(X, b)$ is injective.

5. Consider the topological spaced pictured below, formed from a triangulated rectangle by identifying the two vertices labelled "1". (Imagine identifying two corners of this exam.)

   (a) Write down the simplicial complex obtained from this diagram.

   (b) Show that any edge loop based at 1 is equivalent to an edge loop with vertices in $1, 2, 3$.

   (c) Define the winding number of a loop l to be $w(l)$ = (the number of times (2, 3), (5, 3) or (5, 6) appears in l) $-$ (the number of times (3, 2), (3, 5) or (6, 5) appears in l). Show that equivalent edge loops have the same winding number.

   (d) Show that the fundamental group of X is $\mathbb{Z}$.

   (e) Consider the subcomplex highlighted in red. Explicitly describe the fundamental group of this subcomplex.

   (f) Show that there is no retraction of $|K|$ onto this subcomplex.