Algebraic Topology Final Exam 2018

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This exam is due on Monday 7th May at noon. You are allowed to use the lecture notes by Marc Lackenby and notes made in class. You should not use the internet, and should not discuss with fellow students. Please sign your final exam to state that you have adhered to these guidelines.

You should hand this in via email, directly to me (Office SEC LL010) or the mailbox in the department office. If you choose the latter please email to inform me you have handed it in.

Throughout you may assume that fundamental group S^1 is $\mathbb{Z}$ and the fundamental group of $S^1 \times S^1$ is $\mathbb{Z}^2$.

1. Let $C_{00}(\mathbb{R}) = \{(v_i)_{i\in\mathbb{N}} \mid v_i \in \mathbb{R}, \exists N \text{ s.t. } v_n = 0, \forall n > N\}$ be the set of sequences with finitely many non-zero terms. This space has a norm given by $\|(v_i)\| = \sqrt{\sum_{i=0}^{\infty} v_i^2}$.

   Let S be the subspace $\left\{(v_i) \,\middle|\, \sum_{i=0}^{\infty} v_i^2 = 1\right\}$.

   Let $T\colon S \to S$ be given by $T((v_i)_{i\in\mathbb{N}}) = (w_i)_{i\in\mathbb{N}}$ where $w_i = \begin{cases} 0, i = 0, \\ v_{i-1}, \text{else.} \end{cases}$

   This is the map that shifts every coordinate one to the right.

   (a) [2 marks] Define the straight line homotopy in $C_{00}(\mathbb{R})$.

   (b) [2 marks] Show that the straight line from s to $T(s)$ does not go through the origin for all $s \in S$.

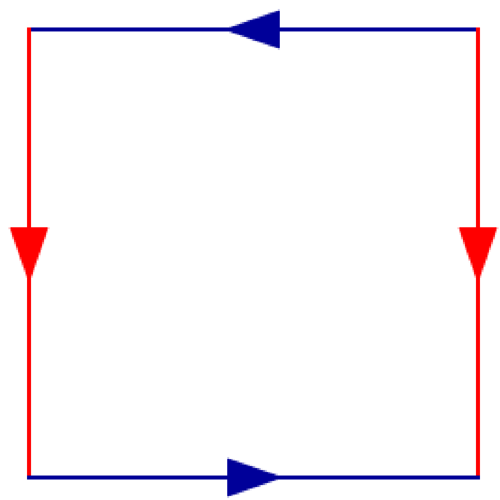
   (c) [4 marks] Show that T is homotopic to the identity map on S.

(d) [3 marks]
Show that $C_{00}(\mathbb{R})$ is contractible.

(e) [4 marks] Deduce that T is homotopic to a constant map.
(You may assume $S \setminus \{(1,0,0,\dots)\}$ is homeomorphic to $C_{00}(\mathbb{R})$.)

(f) [1 marks] Deduce that S is contractible.

2. (a) [2 marks] Let the Klein bottle obtained by identifying sides of a square as below. Explain how this gives a cell structure on the Klein bottle. How many cells are there in each dimension?



(b) [2 marks] Show that the fundamental group of the Klein bottle is $\langle a, b \mid aba^{-1} = b^{-1} \rangle$.

(c) [2 marks] Show that $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ is isomorphic to $\langle c, d \mid c^2 = d^2 \rangle$.

(d) [3 marks] The centre of a group $Z(H) = \{h \in H \mid gh = hg, \forall g \in H\}$
Show that $Z(F(S)) = \{e\}$ if $|S| > 1$.

(e) [2 marks] Show that $c^2 \neq e$ in G.
(Hint: Construct a homomorphism $G \to \mathbb{Z}$.)

(f) [3 marks] Show that $Z(G) \neq \{e\}$.
(Hint: $c^2 \in Z(G)$)

(g) [4 marks] By considering homomorphisms to S_3 show that $cd \neq dc$ in $\langle c, d \mid c^2 = d^2 \rangle$.

(h) [2 marks] Show that the Klein bottle is not homotopy equivalent to a graph or a torus.

3. Let M be the Mobius band, $[-1,1] \times [0,1]/\sim$, where the equivalence is generated by $(x,0) \sim (-x,1)$. Let T be the torus with a disc removed.

(a) [3 marks] State the Seifert-van Kampen theorem.

(b) [5 marks] Compute $\pi_1(M,a)$ and $\pi_1(T,b)$.
Let $i : S^1 \to T$ be the inclusion of the circle as the boundary of the removed disc. Let $j : S^1 \to M$ be the inclusion of the circle as the boundary of the Mobius band.

(c) [7 marks] Compute the images of i_* and j_*.

(d) [5 marks] Compute the fundamental group of the quotient space

$$T \cup M/(i(x) \sim j(x))$$

4. (a) [2 marks] State the Simplicial approximation theorem.

(b) [4 marks] Using the simplicial approximation theorem or otherwise, show that S^n is simply connected for $n > 1$.

(c) [9 marks] Show that any two maps $S^1 \times S^1 \to S^3 \times S^3$ are homotopic. (You may assume that a map $f\colon X \to Y \times Y$ where $f(x) = (f_1(x), f_2(x))$ is continuous if and only if f_1 and f_2 are continuous.)

(d) [5 marks] Let b be a point on S^3. Find two maps $f, g\colon (S^1 \times S^1, (1,1)) \to (S^1 \times S^3, (1,b))$ which are not homotopic relative to $(1,1)$.