Algebraic Topology Sheet 3

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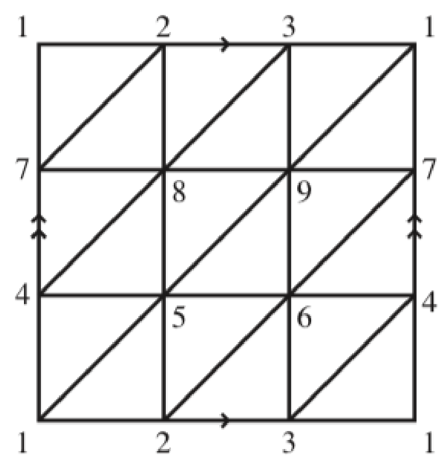
1. Show that $\pi_1(X \times Y, (x, y)) = \pi_1(X, x) \times \pi_1(Y, y)$. (Hint: use the projections $X \times Y \to X$ and $X \times Y \to Y$.)

2. Consider the following triangulation of the torus.



Let x and y be the loops $(1, 2, 3, 1)$ and $(1, 4, 7, 1)$, and let K be the union of these two loops (ie. K comes from the boundary of the square).

(a) Show that any edge path that starts and ends on K but with the remainder of the path missing K is equivalent to an edge path lying entirely in K.

(b) Prove that any edge loop based at 1 is equivalent to an edge loop lying entirely in K.

(c) Deduce that any edge loop based at 1 is equivalent to a word in the alphabet $\{x, y\}$.

(d) Show that the edge loops xy and yx are equivalent.

(e) Deduce that any edge loop based at 1 is equivalent to $x^m y^n$, for $m, n \in \mathbb{Z}$.

(f) Prove that if $x^m y^n \sim x^M y^N$, then $m = M$ and $n = N$. [Hint: define winding numbers as in the proof that $\pi_1(S^1) = \mathbb{Z}$.]

(g) Deduce that the fundamental group of the torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

3. Recall a *retraction* of X onto $A \subset X$ is a map $r \colon X \to A$, such that $r \circ i \colon A \to A$ is the identity map.

   (a) Show that there is no retraction D^2 onto S^1. (Hint: Consider the induced maps on fundamental groups.)

   (b) Show that every map $f \colon D^2 \to D^2$ has a fixed point. To start you off, suppose that $f(x) \neq x$ for all $x \in D^2$. Use the pairs $(x, f(x))$ to construct a retraction D^2 onto S^1, reaching a contradiction. Thus, f must have a fixed point.

   (c) Use similar reasoning to show that any map $f : D^2 \to D^2$ which is the identity on the boundary must be surjective.

4. Show the following:

   (a) $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$, $n \geq 2$.

   (b) $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, $n > 2$.

   Note the following:

   (a) The fundamental group is invariant under homeomorphism.

   (b) Being path connected is invariant under homeomorphism.

5. (Optional) Show that the Mobius strip M does not have a retraction onto its boundary circle.