Algebraic Topology: Problem Sheet 2

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February 14, 2018

1. Let $\alpha : S^n \to S^n$ be the antipodal map ($\alpha(x) = -x$). Prove if n is odd, then $\alpha \simeq id$.

2. Let $f, g : X \to S^n$ be maps satisfying $f(x) + g(x) \neq 0$ for all $x \in X$. Prove $f \simeq g$.

3. Show that any contractible space is path connected.

4. Let X be a contractible space. Show that $X \times Y \simeq_{h.e.} Y$.

5. Let X be a path connected space. Show that the following statements are equivalent:

    (a) X is contractible.

    (b) For any space Y, any 2 maps $f, g : Y \to X$ are homotopic.

    (c) For any path connected space Y, any 2 maps $k, l : X \to Y$ are homotopic.

6. Show that $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

7. Show the following 3 spaces are homotopy equivalent.

    (a) The union of 2 circles with one point identified (A figure 8 graph).

    (b) A torus with a disk removed (cf. Problem sheet 1)

    (c) $\mathbb{R}^2 \setminus \{(-1, 0), (1, 0)\}$.

8. Let $f, g : S^m \to S^n$ be two maps. Prove that if $m < n$, then $f \simeq g$. (Hint: Use the simplicial approximation theorem.)

9. (Optional) Let $f, g : S^{n-1} \to X$ be maps such that $f \simeq g$. Prove $X \cup_f D^n \simeq_{h.e.} X \cup_g D^n$.

10. (Optional) The wedge product of two non-empty spaces $X \vee Y$ is quotient of the disjoint union identifying one point in X with one point in Y. Show that $\mathbb{R}^2 \smallsetminus \{p_1, \ldots, p_k\}$ is homotopy equivalent to $S^1 \vee \cdots \vee S^1$ where there are k copies of S^1.

11. (Doubly optional) Show that $\mathbb{R}^n \smallsetminus \{p_1, \ldots, p_k\}$ is homotopy equivalent to $S^{n-1} \vee \cdots \vee S^{n-1}$ where there are k copies of S^{n-1}.