Solutions for problem sheets

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Sheet 1

1. Add picture
2. Let (V, Σ) be a simplicial complex. The identity map $V \to V$ gives a simplicial map $(V, \Sigma) \to (V, \mathcal{P}(V))$, the latter represents a simplex.

   If we had a smaller triangulation then it would have < 3 vertices. We can now enumerate the possible complexes with < 3 vertices. These are a point, S^0 and an interval.
3. We can see that there are edges which are not uniquely defined by their endpoints.
4. Add pictures. To deduce the cell structures we just count the number of vertices and edges in each picture after identification. Each picture has a single 2-cell.
5. To get a cell structure on the 3-torus. Take a cube and identify faces in pairs.

Sheet 2

1. Recall

   $S^{2n-1} = \{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} : \Sigma x_i^2 = 1\} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \Sigma |z_i|^2 = 1\}.$

   We can thus set up a homotopy $H : S^{2n-1} \times I \to S^{2n-1}$ by

   $$H((z_1, \ldots, z_n), t) = (e^{i\pi t} z_1, \ldots, e^{i\pi t} z_n).$$

We can see that at $t = 0$ this is the identity map and at $t = 1$ this is the antipodal map.

2. Set up a homotopy $H : S^n \times I \to S^n$ as follows.

$$H(x,t) = \frac{tf(x) + (1-t)g(x)}{\|tf(x) + (1-t)g(x)\|}.$$

We must check that this is well defined. This is the same as showing that $\|tf(x) + (1-t)g(x)\| \neq 0$.

$$\begin{aligned} \|tf(x) + (1-t)g(x)\| &= 0 \\ \Rightarrow tf(x) + (1-t)g(x) &= 0 \\ \Rightarrow tf(x) &= (t-1)g(x) \end{aligned}$$

By taking norms of both sides, we see that $t = \frac{1}{2}$

$$\Rightarrow f(x) = -g(x)$$

Thus this homotopy is well defined. And the maps f, g are homotopic.

3. Let b be a point of X and $H : X \times I \to X$ be a homotopy from the identity map on X to the constant map c_b.

   (a) Given a point $x \in X$, $H(x,t)$ gives a path from x to b composing these we can join any two points.

   (b) Let $p : X \times Y \to Y$ and $i : Y \to X \times Y$ be the maps defined by $p(x, y) = y$ and $i(y) = (b, y)$.
   Firstly note that $p \circ i$ is the identity on Y. Thus we are left to prove that $i \circ p$ is homotopic to the identity on $X \times Y$. The homotopy K is given as follows.

$$K((x,y),t) = (H(x,t), y).$$

   We can see that at one end this is the identity and at the other is the map $i \circ p$.

   (c) Let $f, g : Y \to X$ be two maps. We postcompose both with the identity on X noting that the identity is homotopic to the constant map c_b we get the following:

$$f = \mathrm{id} \circ f \simeq f \circ c_b = c_b = g \circ c_b \simeq \mathrm{id} \circ g = g.$$

(d) Let $k, l : X \to Y$ be two maps. We precompose with the identity on X noting that this is homotopic to a constant map and that Y is path connected we get the following:

$$k = k \circ \mathrm{id} \simeq k \circ c_b = c_{k(b)} \simeq c_{l(b)} = l \circ c_b \simeq l \circ \mathrm{id} = k.$$

4. Add pictures

5. Let $f, g : S^m \to S^n$ be two maps and let $m < n$. Both spaces are homeomorphic to simplicial complexes as follows. Let V_i be a set with $i+1$ elements. ^{i} is homeomorphic to $(V_i, \mathcal{P}(V_i) \setminus \{V_i\})$. This is the boundary of an $i - simplex$.

   We can now apply the simplicial approximation theorem to the map f to get a simplicial map h from a subdivision of the simplicial complex. Noting that the triangulation of S^m has simplices of dimension at most m we see that the map $|h|$ is not surjective.

   Thus the image is contained in $S^n \setminus \{*\}$. This is homeomorphic to $\mathbb{R}^n$ and by the previous question is thus homotopic to a constant map. Similar reasoning for g shows that the maps are both homotopic to constant maps and as S^n is path connected these maps are homotopic.

6. Add later

Sheet 3

1. Let $p : X \times Y \to X$ be the map $p(x, y) = x$ and $q : X \times Y \to Y$ be the map $q(x, y) = y$. These are continuous maps, so we get maps $p_* : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x)$ and $q_* : \pi_1(X \times Y, (x, y)) \to \pi_1(Y, y)$.

   Define a map $\phi : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$ $\phi(g) = (p_*(g), q_*(g))$. We must show that this map is surjective and injective.

   Given an element $(h, k) \in pi_1(X, x) \times \pi_1(Y, y)$. Let $\alpha : I \to X$ be a loop representing h and $\beta : I \to Y$ be a loop representing k. The loop $\gamma(t) = (\alpha(t), \beta(t))$ is a loop in $X \times Y$. It is easy to see that $\phi([\gamma]) = (h, k)$. Thus ϕ is surjective.

   To show that it is injective pick an element γ of the kernel of ϕ. Since it is in the kernel $\phi(\gamma) = (p_*(\gamma), q_*(\gamma)) = (e, e)$. Let H be a homotopy showing that $p_*(\gamma)$ is trivial and K be a homotopy showing that $q_*(\gamma)$ is trivial. The map $L : X \times Y \times I \to X \times Y$, defined by $L(a, b, t) = (H(a, t), K(b, t))$. We can see that this is a homotopy between the constant map and γ. Thus ϕ is injective.

2. (a) A retraction $r : D^2 \to S^1$ is a map such that $r \circ i : S^1 \to S^1$ is the identity map. We can now look at the induced maps on fundamental groups noting that $\pi_!(S^1, 1) = \mathbb{Z}$ and $\pi_1(D^2, 1) = \{e\}$. We have the following maps.

$$r_* : \mathbb{Z} \to \{e\}$$

$$i_* : \{e\} \to \mathbb{Z}$$

   such that $r_* \circ i_*$ is the identity on $\mathbb{Z}$ however since it factors through $\{e\}$ it is the trivial map.

   (b) Assume that $f : D^2 \to D^2$ is a map with no fixed points. Construct a map $g : D^2 \to S^1$ as follows. Since $f(x) \neq x$ we can find a real number $s > 0$ such that $\|(1-s)f(x) + sx\|$. Define $g(x)$ to be this point. This is a continuous map which is the identity on the circle. By part a) we see that this map cannot exist and so f must have had a fixed point.

3. If any of these spaces were homeomorphic, then they would also be homeomorphic after removing a point from each. We know that $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to S^n. And by the previous sheet S^n is simply connected for $n > 1$. Thus the results follow.

4. A retraction $r : M \to S^1$ is a map such that $r \circ i : S^1 \to S^1$ is the identity map. We can now look at the induced maps on fundamental groups noting that $\pi_!(S^1, 1) = \mathbb{Z} = \pi_1(M, 1)$. We have the following maps.

$$r_* : \mathbb{Z} \to \mathbb{Z}$$

$$i_* : \mathbb{Z} \to \mathbb{Z}$$

   such that $r_* \circ i_*$ is the identity on $\mathbb{Z}$ however the map i_* is multiplication by 2, this map does not have an inverse.

Sheet 4

1. Let h be a non trivial element of the centre of $F(S)$. It is defined by a reduced word $s_1s_2\dots s_n$. Since S has at least 2 elements we can pick s such that $s \neq s_1, s_1^{-1}$. Thus the word $ss_1s_2\dots s_n$ is reduced and is equal to $s_1s_2\dots s_ns$ since h is in the centre. This second word is either reduced or $s = s_n^{-1}$ and $s_1s_2\dots s_{n-1}$ is reduced.

   The words $ss_1s_2\dots s_n$ and the other reduced word represent the same element but start with different letters. This leads us to a contradiction.

2. Assume that S and T have the same cardinality. Let $f : S \to T$ be a bijection.

   This gives a map $S \to F(T)$ which extends to a group homomorphism $\phi : F(S) \to F(T)$. We can do the same procedure with f^{-1} to get a map $\psi : F(T) \to F(S)$. These maps are mutually inverse.

   For the other direction, we know that by the universal property of free groups that homomorphisms $F(S) \to \mathbb{Z}/2\mathbb{Z}$ are in bijection with maps $S \to \mathbb{Z}/2\mathbb{Z}$. There are $2^{|S|}$ maps $S \to \mathbb{Z}/2\mathbb{Z}$. Thus we see that for the groups to be isomorphic we must have $2^{|S|} = 2^{|T|}$.

3. Add later

4. Add later

5. Add later

Sheet 5

1. A presentation for the product $G_1 \times G_2$ is

$$\mathcal{Q} = \langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup \{xyx^{-1}y^{-1} : x \in X_1, y \in X_2\}\rangle$$

   To prove that this gives the required presentation we need an isomorphism $\phi\colon \mathcal{Q} \to G_1 \times G_2$.

   There is a map $f_1\colon X_1 \sqcup X_2 \to G_1$ given by

$$f_1(x) = x, \forall x \in X_1$$

$$f_1(y) = e, \forall y \in X_2$$

Define $f_2 \colon X_1 \sqcup X_2 \to G_2$ similarly. These extend to homomorphisms $\phi_i \colon \mathcal{Q} \to G_i$ by van Dyk's lemma. This gives us the map $\phi \colon \mathcal{Q} \to G_1 \times G_2$ by

$$\phi(g) = (\phi_1(g), \phi_2(g)).$$

We must check this is surjective and injective.

To prove that it is surjective it is enough to note that it maps onto the set $X_1 \times \{e\} \cup \{e\} \times X_2$, this is clear from the definition of ϕ_i.

To check that it is injective take a word w in the kernel. Using the added relations we can assume that any occurence of a letter of X_1 appears to the left of any letter of X_2. Since in the group $\mathcal{Q}$

$$yx = yx(x^{-1}y^{-1}xy) = xy.$$

Thus we can assume w has the form w_1w_2 where w_i is a word in X_i. Thus the image of this word under ϕ is (w_1, w_2). Since w is in the kernel this must be trivia, i.e. $w_i \in \langle\langle R_i \rangle\rangle$. And so we see that $w_1w_2 \in \langle\langle R_1 \cup R_2 \rangle\rangle$ and the map is injective.

2. The group $G_1 *_{G_0} G_2$ has the presentation

$$\langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) : \forall g \in G_0\} \rangle.$$

We wish to show this is isomorphic to the group given by the presentation

$$\langle X_1 \sqcup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(s) = \phi_2(s) : \forall s \in S\} \rangle.$$

This is equivalent to showing that

$$\langle\langle R_1 \cup R_2 \cup \{\phi_1(s) = \phi_2(s) : \forall s \in S\} \rangle\rangle = \langle\langle R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) : \forall g \in G_0\} \rangle\rangle.$$

One containment is obvious for the other we note that every element of G_0 is a word in S. Given $g = s_1 \dots s_n \in G_0$, we get the following equalities.

$$\phi_1(g) = \phi_2(g)$$
$$\phi_1(s_1 \dots s_n) = \phi_2(s_1 \dots s_n)$$
$$\phi_1(s_1) \dots \phi_1(s_n) = \phi_2(s_1) \dots \phi_2(s_n)$$

This last line is implied the relations $\{\phi_1(s) = \phi_2(s) : \forall s \in S\}$. Thus these two groups are isomorphic.

3. By the previous question the pushout has the presentation

$$\langle x, y \mid y = x^2 \rangle.$$

Using a Tietze transformation we see that this is isomorphic to the group $\langle y \mid \rangle = \mathbb{Z}$.

4. We know that the pushout has the presentation

$$\langle a, b \mid a^2 = b^3 \rangle.$$

We can set up a map to the required group by

$$\begin{aligned} a &\mapsto xyx, \\ b &\mapsto xy. \end{aligned}$$

We must check that $a^2 b^{-3}$ maps to the identity. This is the same as showing that $(xyx)(xyx) = xyxyxy$. By the relation we can replace the second xyx with yxy and we are done.

We can see that $b^{-1}a$ maps to x and $a^{-1}b^2$ maps to y. Thus we can create an inverse map

$$\begin{aligned} x &\mapsto b^{-1}a, \\ y &\mapsto a^{-1}b^2. \end{aligned}$$

We need to check this defines a homomorphism, this is the same as saying that the image of xyx equals the image of yxy. Which is the same as the following

$$\begin{aligned} b^{-1}aa^{-1}b^2b^{-1}a &= a^{-1}b^2b^{-1}aa^{-1}b^2 \\ a &= b^3a^{-1} \end{aligned}$$

This last line is true since $a^2 = b^3$.

This map is inverse to the previous map and as such they are isomorphisms.

5. We set up an isomorphism by

$$c \mapsto ab$$
$$d \mapsto b^{-1}$$

The hint tells us that this will define a homomorphism. The map defined by

$$a \mapsto cd,$$
$$b \mapsto d^{-1},$$

defines an inverse homomorphism.

6. Add later

Sheet 6

1. See pictures.