Covers of a wedge sum

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Apologies for the crude drawings.

Recall that given two spaces X, Y and points $x \in X$ and $y \in Y$. We can define the wedge sum $X \vee Y$ as the quotient of $X \sqcup Y/x \sim y$. We will denote the image of x (or y) under the quotient map b. We identify X and Y with their images under this quotient map.

To get a cover of a wedge sum $X \vee Y$, we can use the following procedure.

Let $\{q_i \colon Y_j \to Y : j \in J\}$ be a collection of covers of Y and $\{p_i \colon X_i \to X : i \in I\}$ be a collection of covers of X, where I and J are finite. Assume that there is a bijection

$$f \colon \bigsqcup_{i \in I} p_i^{-1}(x) \to \bigsqcup_{j \in J} q_j^{-1}(y)$$

.

Then the path components of the space

$$\bigsqcup_{i \in I} X_i \cup \bigsqcup_{j \in J} Y_j$$

is a cover of $X \vee Y$. The covering map r is defined map is defined by applying the covering map p_i or q_j when you are in X_i or Y_j respectively. For what follows we will assume the space

$$\bigsqcup_{i \in I} X_i \cup \bigsqcup_{j \in J} Y_j$$

is path connected.

To check this a cover, we must check what the primage of an open set is. We must prove that for each point there is an open neighbourhood under which the map r^{-1} gives a disjoint union of homeomorphic copies of this neighbourhood.

If the point c is not b then we can find an open set entirely contained in X or Y which contains it. Assume that the point is in X. We can see that the preimage of c will be

$$\sqcup_{i \in I} p_i^{-1}(c).$$

We know that for each map p_i there is an open set with the desired property, the intersection of these sets will be the desired set for r.

We must now consider the point b. We use the above procedure in each side noting that

$$\sqcup_{i \in I} p_i^{-1}(b). = \sqcup_{j \in J} q_j^{-1}(b).$$

We can then take an open map which works for the covering q_j and for the coverings p_i the union of these sets will be the desired set for r.

Consider the following examples.

Let $X = \mathbb{R}P^2$ and $Y = \mathbb{R}P^2$. These spaces are covered by S^2 and these covers have degree 2. We can thus take the quotient of $S^2 \sqcup S^2$ by identifying these pairs of point and this will give a cover of $\mathbb{R}P^2$. See picture for details.

As another consider $X = S^1$ and $Y = \mathbb{R}P^2$. We know that we can construct a cover $S^1 \to S^1$ of degree n for any $n > 0$. We can attach to each point in the preimage of x a copy of $\mathbb{R}P^2$ and this will be a cover of $X \vee Y$. See picture for details.

We can also take a cover of S^1 of degree $2n$ and attach a copy of S^2 to pairs of points giving another cover. See picture for details.

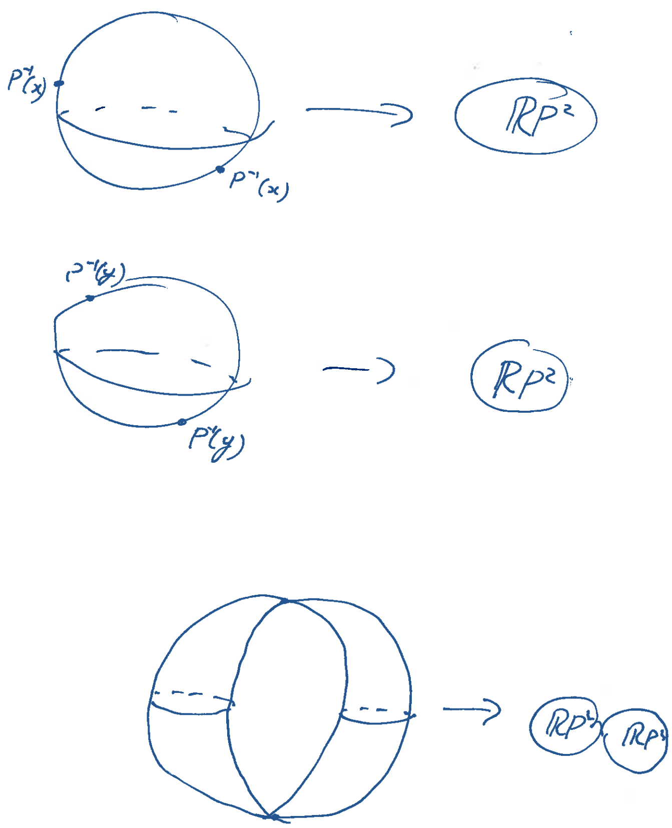

Figure 1: A cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$

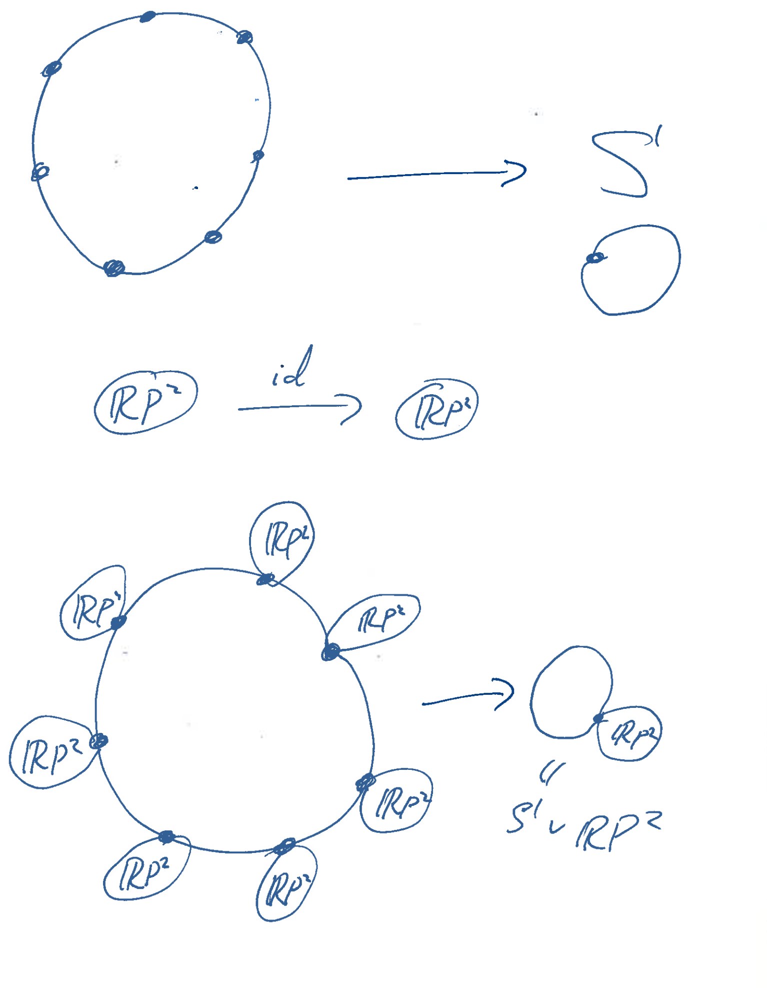

Figure 2: A cover of $\mathbb{R}P^2 \vee S^1$. In this cover the compenents over $\mathbb{R}P^2$ are $\mathbb{R}P^2$

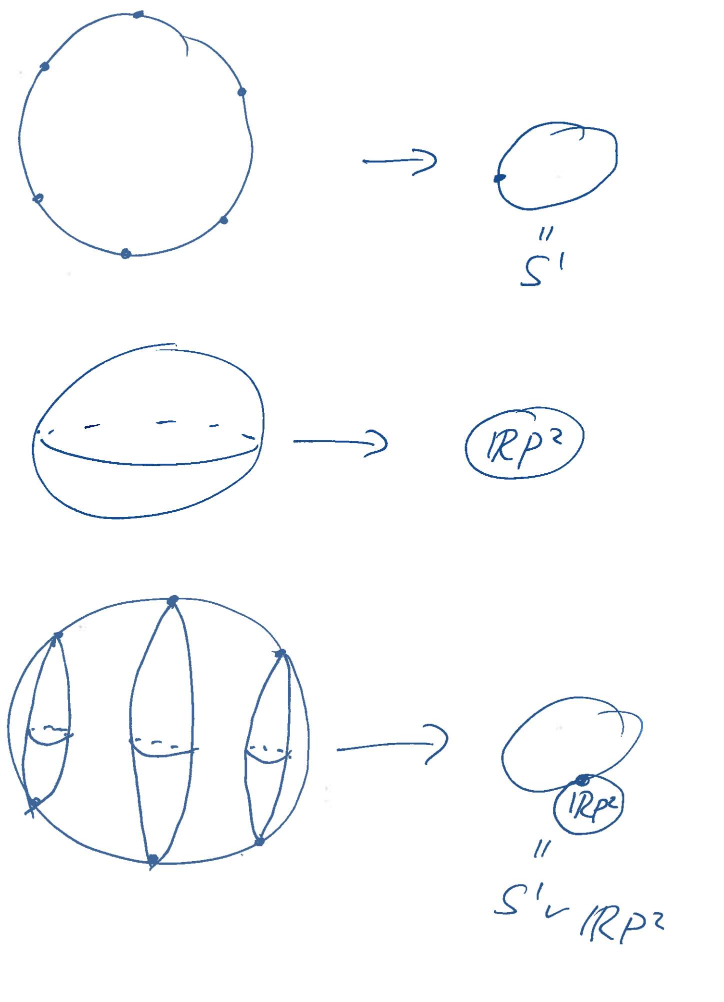

Figure 3: A cover of $\mathbb{R}P^2 \vee S^1$. In this cover the compenents over $\mathbb{R}P^2$ are $\S^2$