Problem Sheet 9

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1. Let $A_1, \ldots, A_n$ be R-modules. Let $B_i \subset A_i$ be submodules. Show that $(A_1 \oplus \cdots \oplus A_n)/(B_1 \oplus \cdots \oplus B_n)$ is isomorphic to $A_1/B_1 \oplus \ldots A_n/B_n$.

2. Let $M = \mathbb{Z}^2$ be the free $\mathbb{Z}$-module of rank 2. Let N be the submodule $24\mathbb{Z} \oplus 30\mathbb{Z}$. Find a basis y_1, y_2 for M and integers a_1, a_2 such that $a_1 \mid a_2$ and $a_1 y_1, a_2 y_2$ is a basis for N. Deduce that $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/30\mathbb{Z}$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$.

3. Let $R = \mathbb{Z}[x]$, let M be the free module of rank 1. Consider the submodule N generated by 2 and x. Show that the rank of N is 1. Show that N is not a free module of rank 1 (This is equivalent to N being a principal ideal).

4. Show that an integral domain R is a PID iff every submodule of a free module is a free module. (Note we proved one direction in class. For the other direction consider cyclic modules.)

5. Let $V = \mathbb{R}^2$ with basis e_1, e_2. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_{\mathbb{R}} V$ cannot be written as any element $v \otimes w$.

6. Let $V = \mathbb{R}^n$ nd let v, v' be non zero elements. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_{\mathbb{R}} V$ if and only if $v = av'$ for some $a \in \mathbb{R}$.

7. Show that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to $\mathbb{C}$.

8. Show that $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}[x]$

9. Show that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is an $\mathbb{R}$-module via $r(c \otimes c') = (rc) \otimes c'$. Show that it is a free R-module of rank 2. Show that the $\mathbb{R}$-module $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has rank > 2.

10. Let R be an integral domain contained in a field Q. Let M be a torsion R module. Show that $M \otimes_R Q = 0$.

11. Let R be an integral domain, Q the field of fractions, M an R-module. Show that every element of $M \otimes_R Q$ can be written as $(1/d) \otimes c$ for some $c \in M, d \in R$.

The following questions are for extra credit!!!

12. Show that $\bigoplus_i = 1^\infty \mathbb{Z}/2^i\mathbb{Z}$ is a torsion $\mathbb{Z}$-module but $\prod_{i=1}^\infty \mathbb{Z}/2^i\mathbb{Z}$ is not a torsion $\mathbb{Z}$-module by showing that the element $(1, 1, \dots)$ has infinite order.

13. Let R be an integral domain and Q the field of fractions. Show that if N is a submodule of an R-module M, then $N \otimes_R Q$ is a submodule of $M \otimes_R Q$.

14. Deduce from the above that tensor product does not commute with direct products. Namely that $(\prod_{i=1}^\infty \mathbb{Z}/2^i\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q} \neq \prod_{i=1}^\infty (\mathbb{Z}/2^i\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q})$