Problem Sheet 8

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1. Show that the ideals (x) and (x, y) are prime ideals of $\mathbb{Q}[x, y]$. Further show that (x, y) is maximal while (x) is not.

2. Let F be a finite field. Show that $F[x]$ has infinitely many prime elements. (Adapt Euclid's proof that $\mathbb{Z}$ has infinitely many primes.)

3. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the polynomials with rational coefficients and integer constant term. This is a subring, you do not have to show this.

   (a) Show that the only units are ± 1.

   (b) Show that the irreducible elements are $\pm p$, for p a prime, and polynomials with constant term 1.

   (c) Show that x cannot be written as a product of irreducible elements. Deduce that R is not a UFD.

   (d) Show that (x) is not a prime ideal.

   (e) (Extra Credit) Describe the ring $R/(x)$.

4. Show that the intersection of two submodules is a submodule and an ascending union of submodules is a submodule.

5. Let M be a module and N a submodule. The *annhilator of* N is the set $\{r \in R \mid rn = 0, \forall n \in N\}$, this is an ideal of R. Let I be an ideal of R. The *annhilator of* I is the set $\{m \in M \mid am = 0, \forall a \in I\}$, this is a submodule of M.

   (a) Let $a_i > 1$. Find the annhilator of the $\mathbb{Z}$-module $\mathbb{Z}/a_1\mathbb{Z} \times \ldots \mathbb{Z}/a_m\mathbb{Z}$.

   (b) Find the annhilator of the ideal $2\mathbb{Z}$ in the above module.

   (c) Let $I \subset R$ be the annhilator of $N \subset M$ show that the annhilator of I contains N. Give an example where they are not equal.

   (d) Let $N \subset M$ be the annhilator of $I \subset R$ show that the annhilator of N contains I. Give an example where they are not equal.

6. Let A be a $\mathbb{Z}$-module, let a be an element of A. Prove that the map $\phi_a \colon \mathbb{Z}/n\mathbb{Z} \to A$ given by $\phi_a(k) = ka$ is a well defined $\mathbb{Z}$-module homomorphism iff $na = 0$. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong \{a \in A \mid na = 0\}$, show this is the annhilator of the ideal $n\mathbb{Z}$.

7. Describe all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/30\mathbb{Z} \to \mathbb{Z}/24\mathbb{Z}$.

8. Let $\text{Tor}(M) = \{m \in M \mid \exists r \in R \text{ such that } rm = 0\}$. Show that if R is an integral domain this is a submodule.

9. Show that if $\phi : M \to N$ is an R-module homomorphism, then $\phi(\text{Tor}(M)) \subset \text{Tor}(N)$.

10. An R-module M is torsion if $M = \text{Tor}(M)$. Show that all finite abelian groups are torsion $\mathbb{Z}$-modules. Find an infinite torsion $\mathbb{Z}$-module.

11. Let R be an integral domain, M a finitely generated torsion module. Show that the annhilator of M is not the zero ideal. Give an example of a not finitely generated torsion module whose annhilator is the zero ideal. (Hint: You can do it over $\mathbb{Z}$.)

12. An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules. Show that M is irreducible iff $M \neq 0$ and M is cyclic with any non-zero element as a generator. Describe all irreducible $\mathbb{Z}$-modules.