Sheet 6 Solutions

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1. Show that any nilpotent element is either 0 or a zero divisor.

   Proof. Let x be a nilpotent element, assume that $x \neq 0$. Let n be the minimal natural number such that $x^n = 0$, by assumption $n > 1$. Then $0 = x^n = x(x^{n-1})$. This shows that x is a zero divisor as long as $x^{n-1} \neq 0$. However $x^{n-1} = 0$ contradicts the minimality of n. □

2. Find all the nilpotent elements of $\mathbb{Z}/60\mathbb{Z}$.

   We are trying to find all the elements x such that there exists and n such that $x^n = 0$. The prime decomposition of 60 is $2^2 \cdot 3 \cdot 5$. If x has prime decomposition $p_1^{i_1} \dots p_k^{i_k}$, then x^n has prime decomposition $p_1^{ni_1} \dots p_k^{ni_k}$. Thus if divisible by 60 we can rearrange to see $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$. Thus $x = 0$ or 30.

3. Removed due to being incorrect.

4. Let r be a unit in R and n be a nilpotent element. Show that $r + n$ is a unit.

   Proof. We will show that $r - n$ has an inverse, this suffices since the nipotents form an ideal by the following proposition and $-(-n) = n$. Let s be such that $rs = 1$. Using the fact that $r^{2k} - n^{2k} = (r - n)(p(r, n))$, where $p(r, n)$ is a polynomial with terms $\pm r^l n^m$. Since there is a k such that $n^{2k} = 0$ we see that $r^2k = r^{2k} - n^{2k} = (r - n)(p(r, n))$. Thus $s^{2k}p(r, n)$ is an inverse for $r - n$. □

5. Show that the set of nilpotent elements $\mathcal{N}(R)$ is an ideal of R.

   Proof. It is clear that this set is closed under multiplication by elements of R since $(rn)^k = r^k n^k$. To show that it is closed under addition we must check that if m and n are nilpotent then $n + m$ is nilpotent. Let k, l be such that $m^k = 0 = n^l$. Then using the binomial theorem we see that $(m + n)^{k+l}$ is 0 since each term has a factor of the form n^i or m^j where $i \geqslant l$ or $j \geqslant k$. □

6. Show that $\mathcal{N}(R/\mathcal{N}) = \{0\}$.

   Proof. Let $n + \mathcal{N}$ be an element of $\mathcal{N}(R/\mathcal{N})$, then $(n+\mathcal{N})^k = \mathcal{N}$ for some k. By definition of multiplication this shows that $n^k + \mathcal{N} = \mathcal{N}$. We know that this means that $n^k \in \mathcal{N}$ but then there exists an l such that $n^{kl} = 0$ i.e. n was already a nilpotent. □

7. Let $\phi \colon R \to S$ be a ring homomorphism. Show that if x is nilpotent $\phi(x)$ is nilpotent.

   Proof. Let x be a nilpotent element and let k be an integer such that $x^k = 0$. Then $0 = \phi(x^k) = \phi(x)^k$ thus $\phi(x)$ is a nilpotent in S. □

8. Assume I is a prime ideal, show that $\mathcal{N}(R) \subset I$.

   Proof. Let x be a nilpotent then, $x^n = 0 \in I$ for some n. If $n = 1$, then $x = 0$ and $x \in I$. If $n > 1$, then $x^n = x^{n-1} \cdot x$ and since I is prime we can see that x or x^{n-1} is an element of I. Repeating in this way we will show that $x \in I$. □

9. Let $\mathrm{rad}(I) = \{x \in R \mid \exists m \in \mathbb{N} \text{ s.t. } x^m \in I\}$. Show that $\mathrm{rad}(I)$ is an ideal of R containing I.

   Proof. It is clear that $I \subset \mathrm{rad}(I)$. Thus we are left to show that it is an ideal. The proof is similar to the proof that nilpotents are an ideal given above. □

10. Show that $\mathcal{N}(R/I) = \mathrm{rad}(I)/I$.

    Proof.

    $$\begin{aligned} \mathcal{N}(R/I) &= \{x \in R/I \mid x^n = 0\} \\ &= \{x \in R/I \mid x^n \in I\} \\ &= \mathrm{rad}(I)/I \end{aligned}$$

    □

11. Show that if I is a prime ideal $\mathrm{rad}(I) = I$.

    Proof. Let x be an element of $\mathrm{rad}(I)$. Then $x^n \in I$, the proof now proceeds in the same way that the nilpotents are in any prime ideal. □

12. Prove the following are equivalent:

    (a) R has one prime ideal.

    (b) $R/\mathcal{N}$ is a field.

(c) Every element of R is either a unit or nilpotent.

Proof. A similar proof was given in class while discussing Zorn's Lemma, details to come shortly. □

References