Algebra 215: Problem Sheet 2

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1. Let $\phi\colon \mathbb{R}^\times \to \mathbb{R}^\times$ be the map sending x to the absolute value of x. Prove that this is a homomorphism, describe the image of ϕ and the kernel of ϕ.
2. The *centre of* G is defined as $Z(G) = \{g \in G \mid gh = hg \forall h \in G\}$. Show that $Z(G)$ is a normal subgroup of G. Prove that if $G/Z(G)$ is cyclic then G is abelian. Hint: If $G/Z(G) = \langle xZ(G) \rangle$. Show that every element of G can be written in the form $x^a z$ for some integer a and some element $z \in Z(G)$.
3. Use the above and Lagrange's theorem to prove that if $|G| = pq$ for primes p, q, then either G is abelian or $Z(G) = \{e\}$
4. The *centraliser of* g is defined as $C_G(g) = \{h \in G \mid hg = gh\}$. Show that $\langle g \rangle \leqslant C_G(g)$ and is in fact a normal subgroup. Show $\bigcap_{g \in G} C_G(g) = Z(G)$.
5. Dummit and Foote pg. 95 q 7, 8
6. Dummit and Foote pg. 101 q 4
7. By induction on $|G|$ show that if G is a finite group then G has a decomposition series.