215 Midterm Fall 2017

This exam is due Monday, October 23 by 12 noon. Please attach this to the front of your work. By signing the bottom of this page you are acknowledging that you did not give or receive help on this exam. You may use your textbook and notes.

Goal: Show that there are exactly 4 groups of order 174.

1. Let $G = H \rtimes_{\varphi} K$.

   Show that if $N \lhd G$ and $N \leqslant H$, then $G/N \cong (H/N) \rtimes_{\bar{\varphi}} K$, where $\bar{\varphi}(k)(hN) = (\varphi(k)(h))N$, you do not have to prove this is an automorphism of H/N.

   Hint: Show $\psi\colon G \to (H/N) \rtimes_{\bar{\varphi}} K, \psi(h,k) = (hN, k)$ is a homomorphism with kernel N.

2. Show that $G = H \rtimes_{\varphi} K$ is isomorphic to $H \times K$ iff φ is trivial.

3. Let p be a prime and let x be a generator of $\mathbb{Z}/p\mathbb{Z}$ the cyclic group of order p.

   Show that the map defined by $\varphi(x) = x^i$ is an automorphism if i is not equivalent to 0 mod p. Deduce that $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ has order $p-1$.

   From now on you may assume that if p is prime, then $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$.

4. Show that if m, n are coprime, then $\mathrm{Aut}(\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}/m\mathbb{Z})\times\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$. (Hint: Show that any automorphism send $(1,0)$ to $(a,0)$.)

5. Let G be a group or order $p^a m$. Assume G has a unique Sylow p subgroup, show that there is a unique quotient of size m.

6. Let G be a group of order 174.

   (a) Show that G has a normal subgroup of order 29, a normal subgroup of order 3 and a subgroup of order 2. Deduce that G has a normal subgroup of order 87.

   (b) Show that any group of order 87 is Abelian. Deduce that there is a unique group of order 87.

   (c) Show that $G = H \rtimes_{\varphi} K$ where $|H| = 87, |K| = 2$.

   (d) Compute the number of maps $K \to \mathrm{Aut}(H)$.

   (e) Show that there are exactly 4 groups of order 174. (One is abelian. For the non-abelian case consider the quotients by Sylow-29 and Sylow-3 subgroups using questions 1 and 5.)

   You may assume there are only two groups of order $2p$ namely, $\mathbb{Z}/2p\mathbb{Z}$ and D_{2p}.